Acta Numerica (2002), pp. 1–71
DOI: 10.1017/S0962492902000014

Structured inverse eigenvalue problems

Moody T. Chu*
Department of Mathematics,
North Carolina State University,
Raleigh, North Carolina, NC 27695-8205, USA
E-mail: chu@math.ncsu.edu

Gene H. Golub†
Department of Computer Science,
Stanford University,
Stanford, California, CA 94305-9025, USA
E-mail: golub@stanford.edu

An inverse eigenvalue problem concerns the reconstruction of a structured matrix from prescribed spectral data. Such an inverse problem arises in many applications where parameters of a certain physical system are to be determined from the knowledge or expectation of its dynamical behaviour. Spectral information is entailed because the dynamical behaviour is often governed by the underlying natural frequencies and normal modes. Structural stipulation is designated because the physical system is often subject to some feasibility constraints. The spectral data involved may consist of complete or only partial information on eigenvalues or eigenvectors. The structure embodied by the matrices can take many forms. The objective of an inverse eigenvalue problem is to construct a matrix that maintains both the specific structure as well as the given spectral property. In this expository paper the emphasis is to provide an overview of the vast scope of this intriguing problem, treating some of its many applications, its mathematical properties, and a variety of numerical techniques.

* This research was supported in part by NSF grants DMS-9803759 and DMS-0073056.
† This work was supported in part by the NSF grant CCR-9971010.

CONTENTS

1. Introduction

In his book *Finite-Dimensional Vector Spaces*, Halmos (1974) wrote:

> Almost every combination of the adjectives proper, latent, characteristic, eigen and secular, with the nouns root, number and value, has been used in the literature for what we call a proper value.

This interesting comment on the nomenclature of eigenvalue echoes the enigmatic yet important role that eigenvalues play in nature. One instance, according to Parlett (1998), is that 'Vibrations are everywhere, and so too are the eigenvalues associated with them.' For that reason, considerable research effort has been expended on eigenvalue computation, especially in the context of matrices. The applications of this research furnish critical insight into the understanding of many vital physical systems.

The process of analysing and deriving the spectral information and, hence, inferring the dynamical behaviour of a system from *a priori* known physical parameters such as mass, length, elasticity, inductance, capacitance, and so on is referred to as a *direct* problem. The *inverse* problem then is to validate, determine, or estimate the parameters of the system according to its observed or expected behaviour. Specifically, in the context of matrices again, an inverse eigenvalue problem (**IEP**) concerns the reconstruction of a matrix from prescribed spectral data.

It is clear that the IEP could trivially be solved if the matrices were subject to no restriction on structure. For the problem to be more significant, either physically or mathematically, it is often necessary to confine the

construction to certain special classes of matrices. Matrices with a specified structure, for example, constitute a special class. Thus an IEP is indeed a *structured* IEP (**SIEP**). The solution to an IEP should satisfy two constraints: the *spectral constraint*, referring to the prescribed spectral data, and the *structural constraint*, referring to the desirable structure. The variation of these constraints defines the variety of IEPs, some of which will be surveyed in this paper.

More should be said about these constraints in order to define an IEP. First we recall one condition under which two geometric entities intersect transversally. Loosely speaking, we may assume that the structural constraint and the spectral constraint define, respectively, smooth manifolds in the space of matrices of a fixed size. If the sum of the dimensions of these two manifolds exceeds the dimension of the ambient space, then under some mild conditions one can argue that the two manifolds must intersect and the IEP must have a solution. A more challenging situation is when the sum of dimensions emerging from both structural and spectral constraints does not add up to the transversal property. In that case, it is much harder to tell whether or not an IEP is solvable. Secondly we note that in a complicated physical system it is not always possible to know the entire spectrum. On the other hand, especially in structural design, it is often demanded that certain eigenvectors should also satisfy some specific conditions. The spectral constraints involved in an IEP, therefore, may consist of complete or only partial information on eigenvalues or eigenvectors. We further observe that, in practice, it may occur that one of the two constraints in an IEP should be enforced more critically than the other, due to the physical realizability, say. Without this, the physical system simply cannot be built. There are also situations when one constraint could be more relaxed than the other, due to the physical uncertainty, say. The uncertainty arises when there is simply no accurate way to measure the spectrum, or no reasonable means to obtain all the information. When the two constraints cannot be satisfied simultaneously, the IEP could be formulated in a least squares setting, in which a decision is made as to which constraint could be compromised.

Associated with any IEP are four fundamental questions. These are issues concerning:

- the theory of *solvability*,
- the practice of *computability*,
- the analysis of *sensitivity*, and
- the reality of *applicability*.

A major effort in solvability has been to determine a necessary or a sufficient condition under which an inverse eigenvalue problem has a solution. The main concern in computability, on the other hand, has been to develop a procedure by which, knowing *a priori* that the given spectral data are

feasible, a matrix can be constructed in a numerically stable fashion. The discussion on sensitivity concerns how the solution to an IEP is modified by changes in the spectral data. The applicability is a matter of differentiation between whether the given data are exact or approximate, complete or incomplete, and whether an exact value or only an estimate of the parameters of the physical system is needed. Each of these four questions is essential but challenging to the understanding of a given IEP. We are not aware of many IEPs that are comprehensively understood in all these four aspects. Rather, considerably more work remains to be done. For the very same reason, we cannot possibly treat each IEP evenly in this article.

With different emphases and different formulations, studies of IEPs have been intensive and scattered, ranging from acquiring a pragmatic solution to a real-world application to discussing the general theory of an abstract formulation. A timely review that better defines the realm of IEPs as a whole is critical for further research and understanding. Earlier endeavours in this regard include the book by Gladwell (1986*b*), where the emphasis was on applied mechanics, the survey by Boley and Golub (1987), where the emphasis was on numerical computation, the book by Zhou and Dai (1991), which pointed to many publications in Chinese that were perhaps unknown to the West, and the article by Gladwell (1996), which reviewed activities and literature between 1985 and 1995 as a ten-year update of his previous book. In a recent review article, Chu (1998) briefly described a collection of thirty-nine IEPs. These problems were categorized roughly according to their characteristics into three types of IEPs, *i.e.*, *parametrized* (**PIEP**), *structured* (**SIEP**), and *partially described* (**PDIEP**). Since then, many more old results have been unearthed, while new articles have continued to appear, notably the treatise by Ikramov and Chugunov (2000), translated from Russian with the emphasis on finitely solvable IEPs and rational algorithms, and the book by Xu (1998), where many results on the sensitivity issue by Chinese mathematicians are made known in English for the first time. It quickly becomes clear that even for SIEPs alone there is a need to update history and describe recent developments in both theory and application. It is for this purpose that this paper is presented.

Although every IEP should be regarded as an SIEP, that view is certainly too broad to be apprehended by a paper of finite length. Thus, our definition of 'structure' is limited to those structures delineated in this paper. Some of these structures, such as Jacobi or Toeplitz, result in matrices forming linear subspaces; some structures, such as nonnegative or stochastic, limit entries of matrices in a certain range; while others, such as matrices with prescribed entries or with prescribed singular values, lead to some implicitly defined structural constraints. We shall touch upon a variety of SIEPs by describing their formulations, highlighting some theories or numerical procedures, and suggesting some relevant references. Additionally, we shall outline some

applications of IEPs from selected areas of disciplines. From time to time, we shall point out some open questions. Let it be noted that, while we sometimes seem to be concentrating on one particular numerical method applied to one particular problem, often the method has enough generality that, with some suitable modifications, it can also be applied to other types of problems. We choose not to encumber readers with the details.

We hope that this presentation, along with previous treatments mentioned above, will help to inspire some additional interest and to stimulate further research that ultimately will lead to a better understanding of this fascinating subject of IEPs.

2. Applications

Inverse eigenvalue problems arise in a remarkable variety of applications. The list includes, but is not limited to, control design, system identification, seismic tomography, principal component analysis, exploration and remote sensing, antenna array processing, geophysics, molecular spectroscopy, particle physics, structural analysis, circuit theory, and mechanical system simulation. In this section we briefly highlight a few applications that, in our judgement, should be of general interest to the readers. So as not to lose sight of the notion of an IEP, it is clear that we have to sacrifice technical details in the description of these applications. We shall divide the discussions into five categories: pole assignment problem, applied mechanics, inverse Sturm–Liouville problem, applied physics, and numerical analysis. Each category covers additional problems.

A common phenomenon that stands out in most of these applications is that the physical parameters of the underlying system are to be reconstructed from knowledge of its dynamical behaviour. The dynamical behaviour is affected by spectral properties in various ways. Vibrations depend on natural frequencies and normal modes, stability controls depend on the location of eigenvalues, and so on. If the physical parameters can be described mathematically in the form of a matrix (as they often are), then we have an IEP. The structure of the matrix is usually inherited from the physical properties of the underlying system.

2.1. Pole assignment problem

Consider first the following dynamic state equation:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \tag{2.1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ denotes the state of a certain physical system to be controlled by the input $\mathbf{u}(t) \in \mathbb{R}^m$. The two given matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are invariant in time. One classical problem in control theory is

to select the input $\mathbf{u}(t)$ so that the dynamics of the resulting $\mathbf{x}(t)$ is driven into a certain desired state. Depending on how the input $\mathbf{u}(t)$ is calculated, there are generally two types of controls, both of which have been extensively studied and documented in the literature.

In the state feedback control, the input $\mathbf{u}(t)$ is selected as a linear function of the current state $\mathbf{x}(t)$, that is,

$$\mathbf{u}(t) = F\mathbf{x}(t). \tag{2.2}$$

In this way, the system (2.1) is changed to a closed-loop dynamical system:

$$\dot{\mathbf{x}}(t) = (A + BF)\mathbf{x}(t). \tag{2.3}$$

A general goal in such a control scheme is to choose the *gain matrix* $F \in \mathbb{R}^{m\times n}$ so as to achieve stability and to speed up response. To accomplish this goal, the problem can be translated into choosing F so as to reassign eigenvalues of the matrix $A + BF$. This type of problem is usually referred to in the literature as a (state feedback) pole assignment problem (**PAP**). It should be pointed out that, in contrast to what we described earlier for an IEP, the matrix F in the context of PAPs does not usually carry any further structure at all. A PAP will become a much harder IEP if F needs to satisfy a certain structural constraint.

It is often the case in practice that the state $\mathbf{x}(t)$ is not directly observable. Instead, only the output $\mathbf{y}(t)$ that is related to $\mathbf{x}(t)$ via

$$\mathbf{y}(t) = C\mathbf{x}(t) \tag{2.4}$$

is available. In the above, $C \in \mathbb{R}^{p\times n}$ is a known matrix. The input $\mathbf{u}(t)$ now must be chosen as a linear function of the current output $\mathbf{y}(t)$, that is,

$$\mathbf{u}(t) = K\mathbf{y}(t). \tag{2.5}$$

The closed-loop dynamical system thus becomes

$$\dot{\mathbf{x}}(t) = (A + BKC)\mathbf{x}(t). \tag{2.6}$$

The goal now is to select the *output matrix* $K \in \mathbb{R}^{m\times p}$ so as to reassign the eigenvalues of $A + BKC$. This output feedback PAP once again gives rise to a special type of IEP (with no constraint on the structure of K).

There is a vast literature of research on the subject of PAPs alone. We would suggest the papers by Byrnes (1989) and by Kautsky, Nichols and Van Dooren (1985), which gave an excellent account of activities in this area as a starting point for further exploration. We shall see later that PAPs are a special case of what we call PIEPs in this article.

One important remark should be made at this point. PAPs, as well as many other IEPs, usually have multiple solutions. Among these multiple solutions, the one that is *least* sensitive to perturbations of problem data is perhaps most critical from a practical point of view. Such a solution,

termed the *robust solution* in the literature, is usually found by minimizing the condition number associated with the solution. In other words, there are two levels of work when solving an IEP for a robust solution: The first is to develop a means to find a solution, if there is any at all; the second is to use optimization techniques to minimize the condition number associated with the solution. Most of the numerical methods discussed in this paper are for the first task only. Except for PAPs (Kautsky *et al.* 1985), the second task for general IEPs has not been fully explored as yet.

For the state feedback problem, there has also been some interest in the case where K is structured. One such application is the so-called *decentralized dynamic system*, where K is a diagonal matrix. Some background information can be found in a recent paper by Ravi, Rosenthal and Wang (1995). Numerical algorithms are needed for this type of structured problems.

2.2. Applied mechanics

Interpreting the word 'vibration' in a broad sense, we see applied mechanics everywhere. The transverse motion of masses on a string, the buckling of structures, the transient current of electric circuits, and the acoustic sound in a tube are just a few instances of vibration. One of the basic problems in classical vibration analysis is to determine the natural frequencies and normal modes of the vibrating body. But inverse problems are concerned with the construction of a model of a given type, for example, a mass-spring system, a string, an IC circuit, and so on, with prescribed spectral data. Such a reconstruction, if possible, would have practical value to applied mechanics and structure design.

Consider the vibration of beads on a taut string illustrated in Figure 2.1. Assume that the beads, each with mass m_i, are placed along the string with equal horizontal spacing h and are subject to a constant horizontal

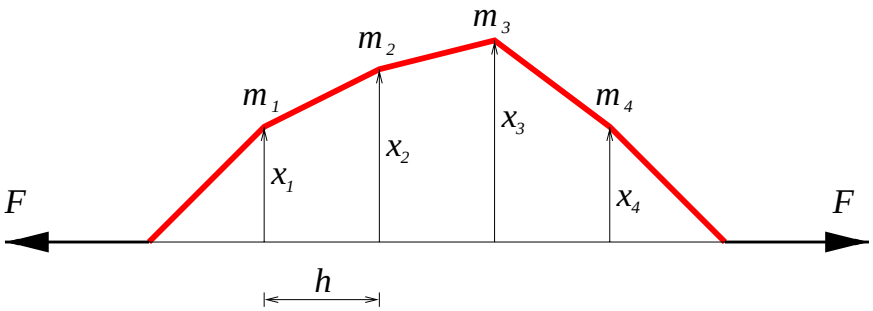

Figure 2.1. Vibration of beads on a string

tension F. Then the equation of motion (for 4 beads) is given by:

$$\begin{aligned}
m_1\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2} &= -F\frac{x_1}{h} + F\frac{x_2-x_1}{h},\\
m_2\frac{\mathrm{d}^2x_2}{\mathrm{d}t^2} &= -F\frac{x_2-x_1}{h} + F\frac{x_3-x_2}{h},\\
m_3\frac{\mathrm{d}^2x_3}{\mathrm{d}t^2} &= -F\frac{x_3-x_2}{h} + F\frac{x_4-x_3}{h},\\
m_4\frac{\mathrm{d}^2x_4}{\mathrm{d}t^2} &= -F\frac{x_4-x_3}{h} - F\frac{x_4}{h}.
\end{aligned}$$

The equation of motion can easily be generalized to the case of n beads, which can conveniently be described in matrix form,

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = -DJ_0\mathbf{x}, \tag{2.7}$$

where $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$, J_0 is the Jacobi matrix

$$J_0 = \begin{bmatrix} 2 & -1 & 0 & & & \\ -1 & 2 & -1 & & & \\ 0 & -1 & 2 & \cdots & & 0 \\ \vdots & & & \ddots & & \\ 0 & & & & 2 & -1 \\ 0 & & & & -1 & 2 \end{bmatrix}, \tag{2.8}$$

and D is the diagonal matrix $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, with $d_i = \frac{F}{m_i h}$. We remark that the system (2.7) may also be thought of the method of lines applied to the one-dimensional wave equation. Eigenvalues of the matrix product DJ_0 are precisely the squares of the so-called *natural frequencies* of the system. An interesting inverse problem that is a special case of the so-called multiplicative IEP (**MIEP**) concerns placing the weights m_i, $i = 1, \ldots, n$ appropriately so that the resulting system has a prescribed set of natural frequencies. An even more fundamental question related to the solvability is whether such a string can have arbitrarily prescribed natural frequencies by adjusting the diagonal matrix D and, if not, what are the reachable frequencies.

More generally, the equation of motion arising in many mechanics applications appears as a linear second-order differential system:

$$M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} = f(\mathbf{x}), \tag{2.9}$$

where $\mathbf{x} \in \mathbb{R}^n$ and M, C, $K \in \mathbb{R}^{n\times n}$. Usually, the *mass matrix* M is diagonal, and both C and the *stiffness matrix* K are symmetric, positive definite, and tridiagonal. It is known that the general solution to the homogeneous equation is a vital prerequisite for the stability of the subsequent

dynamical behaviour. To that end, the fundamental solution can be derived by proposing a solution of the form

$$\mathbf{x}(t) = \mathbf{v}e^{\mu t}.$$

Upon substitution, it turns out that $\mathbf{v}$ and μ are solutions to the quadratic eigenvalue problem

$$(\mu^2 M + \mu C + K)\mathbf{v} = 0. \tag{2.10}$$

Assuming the case that all eigenvalues are distinct, then a general solution to the homogeneous system is given by the superposition principle, that is,

$$\mathbf{x}(t) = \sum_{k=1}^{2n} \alpha_k \mathbf{v}_k e^{\mu_k t},$$

where $(\mu_k, \mathbf{v}_k)$, $k = 1, \ldots, 2n$, are the eigenpair solutions to (2.10).

In the undamped system, where $C = 0$, the quadratic eigenvalue problem is reduced to the generalized eigenvalue problem,

$$(K - \omega^2 M)\mathbf{v} = 0, \tag{2.11}$$

if we write $\lambda = i\omega$. In this case, ω is precisely the natural frequency of the system and $\mathbf{v}$ is the corresponding natural mode. Let $\lambda = \omega^2$, $J := M^{-1/2}KM^{-1/2}$, and $\mathbf{z} = M^{1/2}\mathbf{x}$. The generalized eigenvalue problem can be further reduced to the Jacobi eigenvalue problem

$$J\mathbf{z} = \lambda\mathbf{z}. \tag{2.12}$$

At this point, there are two ways to formulate IEPs in the above context. First, note that the stiffness matrix K is normally more complicated than the mass matrix M. The requirement of maintaining physical feasibility also imposes constraints on the stiffness matrix, making it less flexible and more difficult to construct. Thus, one usual way of forming an IEP is to have the stiffness matrix K determined and fixed from the existing structure, that is, the static constraints, and we want to find the mass matrix M in (2.11) so that some desired natural frequencies are achieved. This inverse problem is equivalent to the MIEP discussed earlier. An alternative formulation is to construct an unreduced, symmetric, and tridiagonal matrix J from its eigenvalues and the eigenvalues of its first leading principal submatrix. This is a special case of the so-called Jacobi IEP (**JIEP**). In Section 4.2, we shall illustrate that such an inverse problem can be identified as configuring a mass-spring system from its spectrum and the spectrum of the same system but with the last mass fixed to have no motion.

The inverse problem for a damped system is considerably more complicated. Assuming that M is normalized to be the identity matrix, the analogous problem to the JIEP for the damped system concerns the reconstruction

of matrices C and K from the given spectral information of the damped system. A particular formulation is given as SIEP6b in Section 4.1.

There are many other types of engineering applications for which an IEP formulation could offer useful insight that, in turn, could lead to better control of performance, safety, or effects of the system. A recent paper by Tisseur and Meerbergen (2001) offers an excellent survey of quadratic eigenvalue problems and related applications. Applications of IEPs to structure design problems can be found in Joseph (1992) as well as the conference collection edited by Mottershead and Friswell (2001). By measuring the changes in the natural frequencies, the IEP idea can be employed to detect the size and location of a blockage in a duct or a crack in a beam. See Wu (1990), Gladwell and Morassi (1999) and Gladwell (1996) for additional references. Studies on IEPs with applications to mechanics are especially flourishing. The research began in the former Soviet Union with the work of M. G. Kreĭn (1933). It first became known in the West through the (German) translation of Gantmaher and Kreĭn (1960). The book by Gladwell (1986*b*) and his follow-up review (Gladwell 1996) cover a broad scope of practices and references of IEPs for small mechanical systems. Applications of IEPs to model updating problems and fault detection problems for machine and structure diagnostics are discussed by Starek and Inman (2001). Other individual articles such as Barcilon (1979), Dai (1995), Gladwell (1984), Gladwell and Gbadeyan (1985), Gladwell (1986*a*, 1997, 1999), Ram and Caldwell (1992) and Ram and Gladwell (1994) represent some typical applications to vibrating rods and beams. A more comprehensive bibliography can be found at our web site `http://www4.ncsu.edu/~mtchu`. Discussion for higher-dimensional problems can be found in Barcilon (1990), Gladwell and Zhu (1992), Knobel and McLaughlin (1994), McLaughlin, Polyakov and Sacks (1994), McLaughlin and Hald (1995) and Zayed (1993). An important extension of the Jacobi-type analysis to a tree-like system is given in Duarte (1989).

2.3. Inverse Sturm–Liouville problems

Much of the discussion of IEPs in the literature has been due to an interest in the inverse Sturm–Liouville problem. A classical regular Sturm–Liouville problem concerns a differential equation of the form:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)+q(x)u(x)=\lambda u(x),\ a<x<b, \tag{2.13}$$

where $p(x)$ and $q(x)$ are piecewise continuous on $[a,b]$ and appropriate boundary conditions are imposed. As a direct problem, it is known that eigenvalues of the system (2.13) are real, simple, countable, and tend to infinity. As an inverse problem, the question is to determine the potential

function $q(x)$ from eigenvalues. This inverse problem has generated much interest in the field, for instance, Andrew (1994), Paine (1984) and Zhornitskaya and Serov (1994), and notably the celebrated work by Gel′fand and Levitan (1955) in which the fundamental fact that two data sequences are required to uniquely determine a potential is settled. A quick introduction to this subject can be found in Chadan, Colton, Päivärinta and Rundell (1997, Chapter 3). A more thoroughgoing discussion was done in the translated book by Levitan (1987).

When a numerical solution is sought, the Sturm–Liouville problem is discretized (Pryce 1993). Likewise, the inverse problem leads to a matrix analogue IEP. Assuming that $p(x) \equiv 1$, $[a, b] = [0, 1]$, and mesh size $h = \frac{1}{n+1}$, the differential equation (2.13) is reduced by the central difference scheme, for instance, to the matrix eigenvalue problem

$$\left(-\frac{1}{h^2}J_0 + X\right)\mathbf{u} = \lambda\mathbf{u}, \tag{2.14}$$

where J_0 is given by (2.8) and X is the diagonal matrix representing the discretization of $q(x)$. The inverse problem is to determine a diagonal matrix X so that the matrix on the left side of (2.14) possesses a prescribed spectrum. This is a special case of the so-called additive IEP (**AIEP**). It should be cautioned that there is a significant difference between the behaviour of the discrete problem and that of the continuous case. See the discussion in Hald (1972) and Osborne (1971). The matrix analogue IEP, such as (2.14), however, is of interest in its own right.

We mention one application to geophysics. Assuming that the Earth has spherical symmetry, geophysicists want to infer its internal structure from the frequencies of spheroidal and torsional modes of oscillations. This leads to the generalized Sturm–Liouville problem, that is,

$$u^{(2k)} - (p_1 u^{(k-1)})^{(k-1)} + \cdots + (-1)^k p_k u = \lambda u.$$

Following the work of Gel′fand and Levitan (1955), Barcilon (1974*a*) suggested that $k + 1$ spectra, associated with $k + 1$ distinct sets of boundary conditions, must be present to construct the unknown coefficients $p_1, \ldots, p_k$. See also Barcilon (1974*b*). It is not clear how the matrix analogue for this high-order problem should be formulated.

2.4. Applied physics

The IEP formulation can sometimes be used to explore and alleviate some difficult computational problems in applied physics. We demonstrate two applications in this section.

We first describe an application to quantum mechanics. In computing the electronic structure of an atom, one usually expands the atom's state

vector over a convenient basis. The expansion coefficients are determined by solving the eigenvalue problem for a Hamiltonian matrix H. It is known that these expansion coefficients are sensitive to the diagonal elements of H. Yet, in many cases of interest, the diagonal elements of H cannot be determined to sufficient accuracy. On the other hand, eigenvalues of H correspond to energy levels of an atom that can usually be measured to a high degree of accuracy. The idea now is to use these measured energy levels to correct diagonal elements. Furthermore, for practical purpose, all matrices involved are required to be real. Under such a constraint, it is almost always impossible to match the eigenvalues exactly. We therefore formulate a least squares IEP (**LSIEP**) as follows (Deakin and Luke 1992). Given a real symmetric matrix A and a set of real values $\boldsymbol{\omega} = [\omega_1, \ldots, \omega_n]^T$, find a real diagonal matrix D such that

$$\|\sigma(A + D) - \boldsymbol{\omega}\|_2$$

is minimized. Throughout this paper, $\sigma(M)$ denotes *either* the spectrum (set) of the matrix M *or* the column vector formed by these eigenvalues: no ambiguity should arise.

We next describe an application to neuron transport theory. One model for the dynamics in an additive neural network is the differential equation

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -A\mathbf{u} + \Omega\mathbf{g}(\mathbf{u}) + \mathbf{p}, \tag{2.15}$$

where $A = \mathrm{diag}(a_1, \ldots a_n)$ denotes the decaying factor, $\Omega = [\omega_{ij}]$ denotes connection coefficients between the neurons, $\mathbf{g}(\mathbf{u}) = [g_1(u_1), \ldots, g_n(u_n)]^T$ denotes the *squashing function*, in which each g_i is strictly increasing but bounded in u_i, and $\mathbf{p}$ is a constant input. One of the design problems is, given A, $\mathbf{g}$, and $\mathbf{p}$, to choose the connection matrix Ω so that a predestined point $\mathbf{u}^* \in \mathbb{R}^n$ is a stable equilibrium. This requirement translates into two conditions that must be satisfied simultaneously. First, the linear equation

$$-A\mathbf{u}^* + \Omega\mathbf{g}(\mathbf{u}^*) + \mathbf{p} = 0 \tag{2.16}$$

must hold for Ω. Secondly, all eigenvalues of the Jacobian matrix,

$$\Upsilon = -A + \Omega G(\mathbf{u}^*), \tag{2.17}$$

where $G(\mathbf{u}^*) = \mathrm{diag}(g_1^{'}(\mathbf{u}^*), \ldots g_n^{'}(\mathbf{u}^*))$, must lie in the left half-plane. Upon rearranging the terms, it is easy to see that (2.16) can be rewritten as

$$\Upsilon\mathbf{x} = \mathbf{y}, \tag{2.18}$$

where $\mathbf{x} = G^{-1}(\mathbf{u}^*)\mathbf{g}(\mathbf{u}^*)$ and $\mathbf{y} = A\mathbf{u}^* - AG^{-1}(\mathbf{u}^*)\mathbf{g}(\mathbf{u}^*) - \mathbf{p}$ are known vectors. This is a special case of the equality constrained IEP (**ECIEP**) considered in Li (1997). Given two sets of real vectors $\{\mathbf{x}_i\}_{i=1}^p$ and $\{\mathbf{y}_i\}_{i=1}^p$ with $p \leq n$, and a set of complex numbers $\{\lambda_1, \ldots, \lambda_n\}$, closed under

conjugation, find a real matrix A such that $A\mathbf{x}_i = \mathbf{y}_i$, $i = 1, \ldots, p$ and $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$. A similar matrix approximation problem with linearly constrained singular values is discussed in Nievergelt (1997).

2.5. Numerical analysis

Finally, we point out that, even within the field of numerical analysis, the notion of IEP helps to shed additional insight on numerical methods and stabilize some numerical algorithms. We comment on four applications: preconditioning, derivation of high-order stable Runge–Kutta schemes, Gaussian quadrature, and low-rank approximations.

Recall first that one of the main ideas in preconditioning a linear equation $Ax = b$ is to transform the original system into an equivalent system that is easier (quicker) to solve with an iterative scheme. The preconditioning of a matrix A can be thought of as implicitly multiplying A by M^{-1}, where M is a matrix for which, hopefully, $Mz = y$ can easily be solved, $M^{-1}A$ is not too far from normal, and $\sigma(M^{-1}A)$ is clustered. This final hope, that the eigenvalues of a preconditioned system $M^{-1}A$ should be clustered, is a loose MIEP criterion. Although, in the context of preconditioning, the locations of eigenvalues need not be exactly specified, the notion of MIEP can certainly help to see what is to be expected of the ideal preconditioner. Many types of unstructured preconditioners have been proposed, including the low-order (coarse-grid) approximation, SOR, incomplete LU factorization, polynomial, and so on. It would be interesting to develop another category of preconditioners where the M is required to possess a certain structure (Forsythe and Straus 1955, Greenbaum and Rodrigue 1989). A related problem that has potential application to optimization is: Given a matrix $C \in \mathbb{R}^{m\times n}$ and a constant vector $\mathbf{b} \in \mathbb{R}^m$, find a vector $\mathbf{x} \in \mathbb{R}^n$ such that the rank-one updated matrix $\mathbf{b}\mathbf{x}^T + C$ has a prescribed set of singular values.

Recall secondly that an s-stage Runge–Kutta method is uniquely determined by the Butcher array

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \ldots & a_{1s} \\ c_2 & a_{21} & a_{22} & \ldots & a_{2s} \\ \vdots & \vdots & & & \vdots \\ c_s & a_{s1} & a_{s2} & \ldots & a_{ss} \\ \hline & b_1 & b_2 & \ldots & b_s \end{array}$$

Let $A = [a_{ij}]$, $\mathbf{b} = [b_1, \ldots, b_s]^T$ and $\mathbf{1} = [1, \ldots 1]^T$. It is well established that the stability function for an s-state Runge–Kutta method is given by

$$R(z) = 1 + z\mathbf{b}^T(I - zA)^{-1}\mathbf{1}$$

(see, for example, Lambert (1991)). To attain numerical stability, implicit methods are preferred. However, fully implicit methods are too expensive. Diagonally implicit methods (**DIRK**), *i.e.*, low triangular A with *identical* diagonal entries, are computationally more efficient, but difficult to construct. As an alternative, it is desirable to develop singly implicit methods (**SIRK**) in which the matrix A does not need to be lower-triangular but must have an s-fold eigenvalue. Such a consideration can be approached by an IEP formulation with prescribed entries, as is done by Müller (1992). Given the number s of stages and the desired order p of the method, define $k = \lfloor (p-1)/2 \rfloor$ and constants $\xi_j = 0.5(4j^2-1)^{-1/2}$, $j = 1, \ldots, k$. Find a real number λ and $Q \in \mathbb{R}^{(s-k)\times(s-k)}$ such that $Q + Q^T$ is positive semi-definite and $\sigma(X) = \{\lambda\}$ where $X \in \mathbb{R}^{s\times s}$ is of the form

$$X = \left[\begin{array}{cc|c} 1/2 & -\xi_1 & \\ \xi_1 & 0 & \\ 0 & & \\ \vdots & \ddots & \\ & 0 & -\xi_k \\ \hline 0 & \xi_k & Q \end{array}\right],$$

and $q_{11} = 0$ if p is even. Note that, in this formulation, the value of the s-fold eigenvalue λ is one of the unknowns to be determined.

Recall thirdly that orthogonal polynomials play a crucial role in the development of Gaussian quadrature rules. Given a weight function $\omega(x) \geq 0$ on $[a, b]$, an n-point Gauss quadrature rule for the integral

$$\mathcal{I}f = \int_a^b \omega(x) f(x)\, \mathrm{d}x \tag{2.19}$$

is a formula of the form

$$\mathcal{G}_n f = \sum_{i=1}^{n} w_i f(\lambda_i), \tag{2.20}$$

with selected nodes $\{\lambda_1, \ldots, \lambda_n\}$ and weights $\{w_1, \ldots, w_n\}$ so that

$$\mathcal{G}_n f = \mathcal{I}f \tag{2.21}$$

for all polynomials $f(x)$ of degree no higher than $2n-1$. With respect to the given $\omega(x)$, a sequence of orthonormal polynomials $\{p_k(x)\}_{k=0}^{\infty}$ satisfying

$$\int_a^b \omega(x) p_i(x) p_j(x)\, \mathrm{d}x = \delta_{ij} \tag{2.22}$$

can be defined. It is an established fact that the roots of each $p_k(x)$ are simple, distinct, and lie in the interval $[a, b]$. Indeed, in order that the

resulting quadrature should achieve the highest degree of precision $2n-1$, the Gaussian nodes should be the roots $\{\lambda_i\}_{i=1}^n$ of $p_n(x)$. On the other hand, it also known that, with $p_0(x) \equiv 1$ and $p_{-1}(x) \equiv 0$, orthogonal polynomials satisfy a three-term recurrence relationship:

$$p_n(x) = (a_n x + b_n)p_{n-1}(x) - c_n p_{n-2}(x). \tag{2.23}$$

Let $\mathbf{p}(x) = [p_0(x), p_1(x), \ldots, p_{n-1}(x)]^T$. This relationship can be written in matrix form as

$$x\mathbf{p}(x) = \underbrace{\begin{bmatrix} \frac{-b_1}{a_1} & \frac{1}{a_1} & 0 & & & 0 \\ \frac{c_2}{a_2} & \frac{-b_2}{a_2} & \frac{1}{a_2} & & & \\ 0 & & & & & \\ \vdots & & & \ddots & \vdots & \\ 0 & & & & & \frac{1}{a_{n-1}} \\ 0 & & & \cdots & \frac{c_n}{a_n} & \frac{-b_n}{a_n} \end{bmatrix}}_{T} \mathbf{p}(x) + \begin{bmatrix} 0 \\ 0 \\ \\ \vdots \\ 0 \\ \frac{1}{a_n}p_n(x) \end{bmatrix}. \tag{2.24}$$

Observe that $p_n(\lambda_j) = 0$ if and only if

$$\lambda_i \mathbf{p}(\lambda_i) = T\mathbf{p}(\lambda_i).$$

Note that the matrix T can be symmetrized by diagonal similarity transformation into a Jacobi matrix J and that the weight w_j in the quadrature is given by

$$w_i = q_{1i}^2, \quad i = 1, \ldots, n,$$

where $\mathbf{q}_i$ is the ith normalized eigenvector of J. This gives rise to an interesting inverse problem. Given a quadrature with nodes $\{\lambda_1, \ldots, \lambda_n\}$ and weights $\{w_1, \ldots, w_n\}$ satisfying $\sum_{i=1}^n w_i = 1$, determine the corresponding orthogonal polynomials (and the corresponding weight function $\omega(x)$; see Kautsky and Elhay (1984), Ferguson (1980)). We illustrate one interesting application to the derivation of the Gauss–Kronrod quadrature rule. Given a Gaussian quadrature (2.20), the associated Gauss–Kronrod quadrature is a $(2n+1)$-point integral rule

$$\mathcal{K}_{2n+1} f = \sum_{i=1}^{n} \tilde{w}_i f(\lambda_i) + \sum_{j=1}^{n+1} \hat{w}_j f(\hat{\lambda}_j) \tag{2.25}$$

that is exact for all polynomials of degree at most $3n+1$. Note that the original nodes $\{\lambda_1, \ldots, \lambda_n\}$ form a subset of the new nodes in $\mathcal{K}_{2n+1}$. Based on an interesting observation in Laurie (1997), the existence of a Gauss–Kronrod quadrature rule with real distinct nodes and positive weights is equivalent to the existence of a real solution to the following special IEP with prescribed entries (**PEIEP**): determine an $n \times n$ symmetric tridiagonal matrix with prescribed first $n-1$ entries (counting row-wise in the upper-

triangular part) and prescribed eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. More details of the computation can be found in the paper by Calvetti, Golub, Gragg and Reichel (2000).

Finally, we note that the problem of low-rank approximation also belongs to the realm of IEPs, considering that a section of the spectrum for the desirable approximation is preset to zero. Low-rank approximation can be used as a tool for noise removal in signal or image processing where the underling matrix is structured as Toeplitz (Cadzow and Wilkes 1990, Suffridge and Hayden 1993), covariance (Li, Stoica and Li 1999, Williams and Johnson 1993), and so on. The rank to be removed corresponds to the noise level where the signal to noise ratio (SNR) is low (Tufts and Shah 1993). Low-rank approximation can also be used for model reduction problems in speech encoding and filter design with Hankel structure, where the rank to be restored is the number of sinusoidal components in the original signal (Park, Zhang and Rosen 1999). The problem of finding or approximating the greatest common divisor (GCD) of multivariate polynomials can be formulated as a low-rank approximation problem with Sylvester structure whose rank is precisely the degree of the GCD (Corless, Gianni, Trager and Watt 1995, Karmarkar and Lakshman 1998). The molecular structure modelling for protein folding in $\mathbb{R}^3$ involves Euclidean distance matrices whose rank is no more than 5 (Glunt, Hayden, Hong and Wells 1990, Gower 1982). In the factor analysis or latent semantic indexing (LSI) application, the low rank is the number of principal factors capturing the random nature of the indexing matrix (Horst 1965, Zha and Zhang 1999). All of these can be considered as structured IEPs with partial spectrum identically zero.

We have seen from the above illustrations that different applications lead to different IEP formulations. We conclude this section with one additional remark by Gladwell (1996), who suggested that, for application purposes, there should also be a distinction between *determination* and *estimation* in the nature of an inverse problem. When the given data are exact and complete, so that the system can be precisely determined, the IEP is said to be *essentially mathematical*. In contrast, we say that we have an *essentially engineering* IEP when the data are only approximate and often incomplete, in which only an estimate of the parameters of the system is sought and the resulting behaviour is expected to agree only approximately with the prescribed data.

3. Nomenclature

For the ease of identifying the characteristics of various IEPs, we have suggested using a unified name scheme ***IEP#** to categorize an IEP (Chu 1998). When singular values are involved in the spectral constraint, we distinguish ISVPs from IEPs. Letter(s) '$*$' in front of IEP register the type

Table 3.1. Summary of acronyms used in the paper

Acronym	Meaning	Reference
AIEP	Additive IEP	Section 9.3
ECIEP	Equality Constrained IEP	Page 12
ISEP	Inverse Singular/Eigenvalue Problem	Section 11
ISVP	Inverse Singular Value Problem	Section 10
JIEP	Jacobi IEP	Section 4
LSIEP	Least Squares IEP	Page 12
MIEP	Multiplicative IEP	Page 8
MVIEP	Multi-Variate IEP	
NIEP	Nonnegative IEP	Section 6
PAP	Pole Assignment Problem	Page 6
PEIEP	IEP with Prescribed Entries	Section 9
PIEP	Parametrized IEP	Page 51
PDIEP	Partially Described IEP	Page 25
RNIEP	Real-valued Nonnegative IEP	Page 39
SHIEP	Schur–Horn IEP	Page 47
SIEP	Structured IEP	Page 3
SNIEP	Symmetric Nonnegative IEP	Page 39
StIEP	Stochastic IEP	Section 7
STISVP	Sing–Thompson ISVP	Page 49
ToIEP	Toeplitz IEP	Section 5
UHIEP	Unitary Hessenberg IEP	Section 8

of problem. The numeral '#' following IEP, if any, indicates the sequence of variation within type '*IEP'. For convenience of later reference, we summarize the acronyms appearing in this paper in Table 3.1. Also indicated are the page numbers or the section numbers where the problems are first described or where more detailed discussion can be found.

Figure 3.1 depicts a possible inclusion relationship between different problems. In particular, the diagram is intended to imply the following.

- Multivariate IEPs include univariate IEPs as a special case.
- All problems have a natural generalization to a least squares formulation.
- The structural constraints involved in SIEPs can appear in various forms, and hence define different IEPs.
- There is a counterpart ISVP corresponding to any structured IEP, formed by replacing the eigenvalue constraint by a singular value constraint. Very little is known about these structured ISVPs (and hence no diagrams).

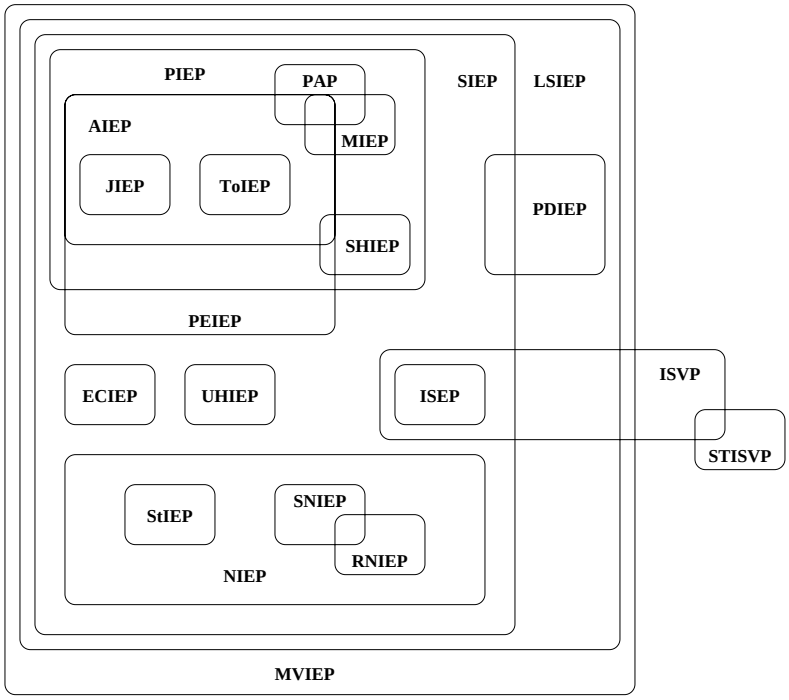

Figure 3.1. Classification of inverse eigenvalue problems

- The class of PIEPs is considered to be a subset of general SIEPs, while many classical IEPs are special cases of PIEPs.
- The relationship depicted in Figure 3.1 is not necessarily definitive because many characteristics may overlap.

This classification along with review articles by Gladwell (1986*c*, 1996), who differentiates problems according to the type of mechanical system, *i.e.*, continuous or discrete, damped or undamped, and the type of prescribed data, *i.e.*, spectral, modal, or nodal, complete or incomplete, should complement each other to offer a fairly broad view of research activities in this area.

This paper concentrates on the SIEP segment only. Even so, the formations and algorithms differ noticeably from problem to problem. Indeed, we pointed out earlier that every IEP should in fact be regarded as an SIEP because of the presence of its structural constraint. That view is too broad to be covered here. Instead, we shall focus on eight selected special structures. These are the IEPs for Jacobi matrices, Toeplitz matrices, nonnegative matrices, stochastic matrices, unitary matrices, matrices with prescribed entries, matrices with prescribed singular values, and matrices with prescribed singular values and eigenvalues. Our criteria of selection

are simply that these eight problems are representative of a variety of structural constraints and are slightly better studied in the literature. We choose not to include PAPs because that topic has been well considered in many other places.

We shall consider these eight structured problems in slightly more breadth and depth with regard to the four issues of solvability, computability, sensitivity, and applicability. Some main results, applications, and algorithmic issues will also be presented.

4. Jacobi inverse eigenvalue problems

By a Jacobi structure, we mean a symmetric, tridiagonal matrix of the form

$$J = \begin{bmatrix} a_1 & b_1 & 0 & & & 0 \\ b_1 & a_2 & b_2 & & & 0 \\ 0 & b_2 & a_3 & & & 0 \\ \vdots & & & \ddots & & \\ & & & & a_{n-1} & b_{n-1} \\ 0 & & & & b_{n-1} & a_n \end{bmatrix}, \tag{4.1}$$

with positive subdiagonal elements $b_i > 0$. We have already seen that this structure arises in many important areas of applications, including oscillatory mass-spring systems, composite pendulum, and Sturm–Liouville problems. Eigenvalues of a Jacobi matrix are necessarily real and distinct. Since J is characterized by the $2n-1$ unknown entries, $\{a_i\}_{i=1}^{n}$ and $\{b_j\}_{j=1}^{n-1}$, it is intuitively true that $2n-1$ pieces of information are needed to solve the inverse problems. That is, to fully describe a JIEP we need additional information other than just the spectrum of J. This additional information comes from different sources and defines additional structures for JIEPs. We shall survey a few JIEPs in this section. One unique and important feature for JIEPs is that often the inverse problem can be solved by direct methods in finitely many steps.

Jacobi matrices enjoy many nice properties. These properties make the study of JIEPs more complete and fruitful than other IEPs. For that reason, we shall provide somewhat more details on the theory and development of JIEPs. We shall touch upon all four fundamental questions raised in Section 1 for JIEPs. We hope that this exertion can serve as a study guide for further developments of other IEPs in the future.

Before we move on, we should emphasize that the JIEPs under discussion here are of tridiagonal structure only. The generalization to band matrices is possible. Some initial studies of the IEP for band matrices can be found in the paper by Biegler-König (1981*a*). Boley and Golub (1987) generalized some of the numerical methods for JIEPs to the banded case. However,

be aware that there are some fundamental differences in the generalization. For instance, two sets of eigenvalues generally determine a tridiagonal matrix uniquely (See Theorem 4.1 in Section 4.3), whereas three sets of eigenvalues do not give a pentadiagonal matrix uniquely (and, in fact, sometimes there is a continuum of solutions) (Boley and Golub 1987).

4.1. Variations

There are several variations in formulating a JIEP. Each formulation can be associated with a mass-spring system. In this section, we only describe the setup, a brief history and some relevant references on the original settings for each problem. Topics on physical interpretation, mathematical theory, and computational methods will be discussed in the next few sections.

In the following, J_k denotes the $k \times k$ principal submatrix of J, and J_{n-1} is abbreviated as $\bar{J}$. Whenever possible, we refer to each variation by the identification name used in Chu (1998).

SIEP6a. Given real scalars $\{\lambda_1, \ldots, \lambda_n\}$ and $\{\mu_1, \ldots, \mu_{n-1}\}$ satisfying the interlacing property

$$\lambda_i < \mu_i < \lambda_{i+1}, \quad i = 1, \ldots, n-1, \tag{4.2}$$

find a Jacobi matrix J such that

$$\begin{cases} \sigma(J) = \{\lambda_1, \ldots, \lambda_n\}, \\ \sigma(\bar{J}) = \{\mu_1, \ldots, \mu_{n-1}\}. \end{cases}$$

This problem is perhaps the most fundamental and extensively studied IEP in the literature. It appears that the problem was originally proposed by Hochstadt (1967), although Downing and Householder (1956) had formulated a more general inverse characteristic value problem much earlier. Much of the existence theory and continuous dependence of the solution on data were developed later in Hochstadt (1974), Gray and Wilson (1976), and Hald (1976). Dangerously many numerical methods are available! Some are stable and some are subtly unstable. We shall discuss some of these methods later. For the time being, it suffices to mention some important works in this regard (Boley and Golub 1987, de Boor and Golub 1978, Erra and Philippe 1997, Gragg and Harrod 1984, Hochstadt 1979, Parlett 1998).

SIEP2. Given real scalars $\{\lambda_1, \ldots, \lambda_n\}$, find a Jacobi matrix J such that

$$\begin{cases} \sigma(J) = \{\lambda_1, \ldots, \lambda_n\}, \\ a_i = a_{n+1-i}, \\ b_i = b_{n+2-i}. \end{cases}$$

Let $Xi \in \mathbb{R}^{n\times n}$ denote the unit perdiagonal matrix where

$$\xi_{ij} = \begin{cases} 1, & \text{if } i = n+1-j, \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

A matrix M is said to be persymmetric if and only if $\Xi M \Xi = M^T$. In other words, the entries of M are symmetric with respect to the northeast-to-southwest diagonal. A persymmetric Jacobi matrix involves only n independent entries. The spectral constraint therefore requires spectrum information only. This problem was first considered in Hochstadt (1967) and then in Hald (1976). Numerical methods for SIEP2 usually come along with those for SIEP6a with appropriate modifications (de Boor and Golub 1978, Parlett 1998).

SIEP7. Given real scalars $\{\lambda_1, \ldots, \lambda_n\}$ and $\{\mu_1, \ldots, \mu_{n-1}\}$ satisfying the interlacing property

$$\begin{cases} \lambda_i \le \mu_i \le \lambda_{i+1}, \\ \mu_i < \mu_{i+1}, \end{cases} \qquad i = 1, \ldots, n-1, \tag{4.4}$$

and a positive number β, find a periodic Jacobi matrix J of the form

$$J = \begin{bmatrix} a_1 & b_1 & & & & b_n \\ b_1 & a_2 & b_2 & & & 0 \\ 0 & b_2 & a_3 & & & 0 \\ \vdots & & & \ddots & & \\ & & & & a_{n-1} & b_{n-1} \\ b_n & & & & b_{n-1} & a_n \end{bmatrix}$$

such that

$$\begin{cases} \sigma(J) = \{\lambda_1, \ldots, \lambda_n\}, \\ \sigma(\bar{J}) = \{\mu_1, \ldots, \mu_{n-1}\}, \\ \prod_1^n b_i = \beta. \end{cases}$$

A periodic Jacobi matrix differs from a Jacobi matrix in that its eigenvalues need not be strictly separated. The interlacing property (4.4) therefore differs from (4.2) in that equalities are allowed. The notion of periodic Jacobi matrices arise in applications such as periodic Toda lattices or continued fractions (Adler, Haine and van Moerbeke 1993, Andrea and Berry 1992). Spectral properties of the periodic Jacobi matrices were first analysed by Ferguson (1980) using a discrete version of Floquet theory, but numerical methods had been proposed earlier in Boley and Golub (1978). See also discussions in Boley and Golub (1984, 1987).

SIEP8. Given real scalars $\{\lambda_1, \ldots, \lambda_n\}$ and $\{\mu_1, \ldots, \mu_n\}$ satisfying the interlacing property

$$\lambda_i < \mu_i < \lambda_{i+1}, \quad i = 1, \ldots, n,$$

with $\lambda_{n+1} = \infty$, find Jacobi matrices J and $\tilde{J}$ so that

$$\begin{cases} \sigma(J) = \{\lambda_1, \ldots, \lambda_n\}, \\ \sigma(\tilde{J}) = \{\mu_1, \ldots, \mu_n\}, \\ J - \tilde{J} \neq 0, \quad \text{only at the } (n, n) \text{ position.} \end{cases}$$

This problem originally appeared in de Boor and Golub (1978). Note that $\tilde{J}$ is a special rank-one update of J. This problem is closely related to SIEP6a in that the theory and numerical methods for SIEP6a will work almost identically for SIEP8. A similar problem involving the preconditioning of a matrix by a rank-one matrix was mentioned earlier in Section 2.5. An application of rank-one updating involving the inverse quadratic eigenvalue problem was discussed in Datta, Elhay and Ram (1997) and Ram (1995).

SIEP9. Given distinct real scalars $\{\lambda_1, \ldots, \lambda_{2n}\}$ and an $n \times n$ Jacobi matrix $\tilde{J}$, find a $2n \times 2n$ Jacobi matrix J so that

$$\begin{cases} \sigma(J) = \{\lambda_1, \ldots, \lambda_{2n}\}, \\ J_n = \tilde{J}. \end{cases}$$

This problem, first discussed in Hochstadt (1979), corresponds exactly to the problem of computing the Gaussian quadrature of order $2n$ that has degree of precision $4n - 1$, given the Gaussian quadrature of order n that has degree of precision $2n - 1$. Several numerical algorithms are available. See Boley and Golub (1987). An IEP as such is actually a special case of a more general category of IEPs with prescribed entries. The latter, in turn, is a subset of so-called completion problems in the literature. The prescribed entries need not be in a diagonal block as in SIEP9. An interesting question related to the IEP is to find the largest permissible cardinality of the prescribed entries so that the completed matrix has a prescribed spectrum. The first publication devoted to this problem was probably due to London and Minc (1972), followed by the series of work by de Oliveira (1973*a*, 1973*b*, 1975). A most recent and comprehensive survey on this topic was given by Ikramov and Chugunov (2000), who stated that the thesis by Hershkowits (1983) contained the strongest result in this class of problems. Also presented in Ikramov and Chugunov (2000) was a careful treatment showing how the completion problems can be solved by finite rational algorithms. A similar inverse problem for matrices with prescribed entries and characteristic polynomial was considered by Dias da Silva (1974); for matrices with prescribed characteristic polynomial and principal submatrices by Silva (1987*a*); and

for matrices with prescribed spectrum and principal submatrices by Silva (1987*b*).

SIEP6b. Given complex scalars $\{\lambda_1, \ldots, \lambda_{2n}\}$ and $\{\mu_1, \ldots, \mu_{2n-2}\}$, distinct and closed under complex conjugation, find tridiagonal symmetric matrices C and K for the λ-matrix $Q(\lambda) = \lambda^2 I + \lambda C + K$ so that

$$\begin{cases} \sigma(Q) = \{\lambda_1, \ldots, \lambda_{2n}\}, \\ \sigma(\bar{Q}) = \{\mu_1, \ldots, \mu_{2n-2}\}. \end{cases}$$

Clearly, SIEP6b is an analogy of SIEP6a applied to a damped system. Strictly speaking, to maintain the physical feasibility a practical solution imposes additional conditions on K and C, that is, both matrices are supposed to have positive diagonal entries, negative off-diagonal entries, and be weakly diagonally dominant. The setup of SIEP6b, where two sets of eigenvalues are given, was considered by Ram and Elhay (1996). Similar inverse problems with prescribed eigenvalues and eigenvectors were studied in a series of works (Starek, Inman and Kress 1992, Starek and Inman 1997, Starek and Inman 2001).

4.2. Physical interpretations

The JIEPs described above can be related to various physical systems, for instance a vibrating beam or rod (Gladwell 1986*b*), a composite pendulum (Hald 1976), or a string with beads (Hochstadt 1967). Correspondingly, the quantities to be determined in a JIEP represent different physical parameters, for instance the stress, the mass, the length, and so on. In this section, we shall use a serially linked, undamped mass-spring system with n particles to demonstrate the physical interpretation of JIEPs. The physical system is depicted in Figure 4.1.

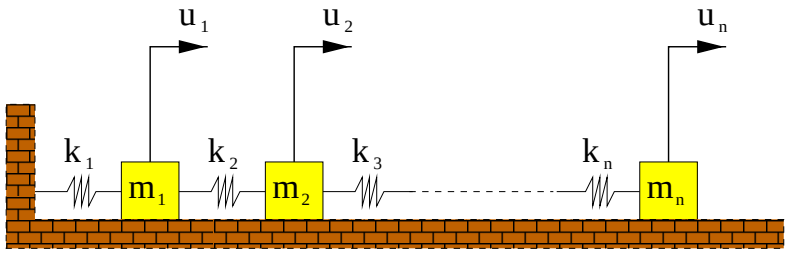

Figure 4.1. Mass-spring system

Suppose that the ith particle has mass m_i, that the springs satisfy Hooke's law, and that the ith spring has spring constant k_i. Let $u_i(t)$ denote the

horizontal displacement of the ith particle at time t. Then it is easy to see that the equation of motion is given by

$$\begin{aligned} m_1 \frac{\mathrm{d}^2 u_1}{\mathrm{d}t} &= -k_1 u_1 + k_2(u_2 - u_1), \\ m_i \frac{\mathrm{d}^2 u_i}{\mathrm{d}t} &= -k_i(u_i - u_{i-1}) + k_{i+1}(u_{i+1} - u_i), \quad i = 2, \ldots, n-1, \\ m_n \frac{\mathrm{d}^2 u_n}{\mathrm{d}t} &= -k_n(u_n - u_{n-1}). \end{aligned}$$

In matrix form, we have

$$M \frac{\mathrm{d}^2 \mathbf{u}}{\mathrm{d}t} = -K\mathbf{u}, \tag{4.5}$$

where $\mathbf{u} = [u_1, \ldots, u_n]^T$, $M = \mathrm{diag}(m_1, \ldots, m_n)$, and K is the Jacobi matrix given by

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \ldots & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & & & 0 \\ 0 & -k_3 & k_3 + k_4 & & & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & & & & & -k_n \\ 0 & & & & -k_n & k_n \end{bmatrix}.$$

A fundamental solution of the form $\mathbf{u}(t) = e^{i\omega t}\mathbf{v}$ leads to the generalized eigenvalue problem (2.11). A transformation $J = M^{-1/2}KM^{-1}$, $\mathbf{z} = M^{1/2}\mathbf{v}$, and $\lambda = \omega^2$ leads to the Jacobi eigenvalue problem (2.12). The direct problem calculates the natural frequencies and modes of the mass-sprint system from given values of m_i and k_k. The inverse problem requires calculating quantities such as $\frac{k_i + k_{i+1}}{m_i}$ and $\frac{k_{i+1}}{\sqrt{m_i m_{i+1}}}$ from the spectral data. Based on this model, we make the following observations.

If the last mass m_n is fastened to the floor, then the motion of mass m_{n-1} is governed by

$$m_{n-1} \frac{\mathrm{d}^2 u_{n-1}}{\mathrm{d}t} = -k_{n-1}(u_{n-1} - u_{n-2}) + k_n(-u_{n-1}),$$

instead. In matrix form the equation of motion for the first $n-1$ particles corresponds exactly to that of deleting the last row and the last column from (4.5). Thus solving SIEP6a is equivalent to identifying the mass-spring system in Figure 4.1 from its spectrum, and from the spectrum of the reduced system where the last mass is held to have no motion. The recovery of the spring stiffness and the masses from the matrix J is discussed in Gladwell (1986*b*).

Likewise, if another spring from m_n is attached to a wall on the far right side of the system, then the equation of motion for m_n is modified to become

$$m_n \frac{\mathrm{d}^2 u_n}{\mathrm{d}t} = -k_n(u_n - u_{n-1}) + k_{n+1}(-u_n).$$

SIEP2 corresponds to the construction of such a mass-spring system from its spectrum if all parameters m_i and k_i are known *a priori* to be symmetric with respect to the centre of the system.

It is a little bit more complicated to sketch a diagram for the physical layout of SIEP7. Basically, we imagine that masses m_1 and m_n are somehow connected by another spring mechanism so as to form a loop (such as the periodic Toda lattice discussed in the literature). Any displacement in either particle of m_1 or m_n will affect each other via that mechanism, contributing nonzero but equal entries at the $(1, n)$ and $(n, 1)$ positions of K, respectively. Apart from this extra connection, the meaning of SIEP7 is now the same as that of SIEP6a.

We can also identify a mass system from its spectrum and from the spectrum of a new system by replacing the last mass and spring with new parameters $\tilde{m}_n$ and $\tilde{k}_n$ satisfying the relationship

$$\frac{k_n^2}{m_n} = \frac{\tilde{k}_n^2}{\tilde{m}_n}.$$

The resulting inverse problem is precisely SIEP8.

The interpretation of SIEP9 is straightforward. It means completing the construction of a mass-spring system of size $2n$ from its spectrum and from existing physical parameters m_i, k_i of the first half of the particles.

Thus far, we have assumed that the system in Figure 4.1 has no friction. For a damped system, the damping matrix C will be part of the parameters and we shall face a quadratic eigenvalue problem (2.10). Other than this, the physical interpretation for each of the JIEPs described above can be extended to damped systems. For example, SIEP6b is to identify the damped system, including its damper configurations, from its spectrum and from the spectrum of the reduced system where the last mass is held immobile. This problem is still open. The principal difficulty is to find conditions on the (complex) spectra which ensure a realistic solution.

It is important to point out that thus far we have considered using only eigenvalues to construct Jacobi matrices. For large and complex systems, it is often practically impossible to gather the entire spectrum information for reconstruction. Partial information with some from eigenvalues and some from eigenvectors can also be used to determine a Jacobi matrix. This type of problem is referred to as PDIEP in Chu (1998) and is beyond the scope of the present paper.

An interesting question related to PDIEPs is how much eigenvector in-

formation is needed to determine such a Jacobi (or any structured) matrix. Gladwell (1996) offered an account from engineering perspectives on why using low-frequency normal modes is important in practice. Some discussion can be found in the books by Zhou and Dai (1991) and Xu (1998). Applications of inverse problems with given spectral and modal data were studied in Gladwell (1986*a*), Starek *et al.* (1992), Starek and Inman (2001), and the many references cited in the review paper by Gladwell (1996). A number of interesting variants of JIEPs may be found in Nylen and Uhlig (1997*a*, 1997*b*), and some corresponding damped problems in Nylen (1999), Gladwell (2001) and Foltete, Gladwell and Lallement (2001).

4.3. Existence theory

Among all IEPs, the class of JIEPs probably enjoys the most satisfactory solvability theory. Most of the existence proofs are based on a recurrence relationship among the characteristic polynomials. More precisely, let $p_k(t) = \det(tI - J_k)$ denote the characteristic polynomial of the leading $k \times k$ principal submatrix J_k. Then

$$p_k(t) = (t - a_k)p_{k-1}(t) - b_{k-1}^2 p_{k-2}(t), \quad k = 2, \ldots, n, \tag{4.6}$$

if $p_0 \equiv 1$. Such a recurrence relationship in fact gives rise to a constructive proof that, in turn, can be implemented as a numerical algorithm. Because there is an extensive literature in this regard, and because some of the constructions will be discussed as numerical methods, we shall only state the existence theorems without proof in this section.

Theorem 4.1. Suppose that all the given eigenvalues are distinct. Then:

(1) SIEP6a has a unique solution (Hald 1976);
(2) SIEP2 has a unique solution (Hald 1976);
(3) SIEP8 has a unique solution (de Boor and Golub 1978).

It should be noted that the MIEP (of uniformly spaced beads on a taut string) described on page 8 is very different from the JIEPs described above in several aspects. The former involves only one *single spectrum*; the latter involves *two spectra*. In the former, we have only one set of parameters (the masses) to adjust; in the latter, we have two sets (the m_is and the k_is) to combine. The solution for the latter is often unique while the former is a much harder problem.

Theorem 4.2. (Ram and Elhay 1996) Over the complex field $\mathbb{C}$, suppose that all the given eigenvalues are distinct. Then SIEP6b is solvable and has at most $2^n(2n-3)!/(n-2)!$ different solutions. In the event that there are common eigenvalues, then there are infinitely many solutions for SIEP6b.

We stress that the solvability of SIEP6b established in the above theorem is over the algebraically closed field $\mathbb{C}$. It is not known whether the problem is realistically solvable with positive masses, springs, and dampers (Gladwell 2001).

Theorem 4.3. (Xu 1998) SIEP7 is solvable if and only if

$$\prod_{k=1}^{n} |\mu_j - \lambda_k| \geq 2\beta(1 + (-1)^{n-j+1}),$$

for all $j = 1, \ldots, n-1$. Even in the case of existence, no uniqueness can be ascertained.

It is worth mentioning that Ferguson (1980) characterized periodic Jacobi matrices by a notion of 'compatible' data that can be turned into a numerical algorithm. Each set of compatible data uniquely determines a periodic Jacobi matrix. On the other hand, these sets of eigenvalues $\lambda = \{1, 3, 5\}$ and $\mu = \{2, 4\}$ with $\beta = 1$ server as a counterexample showing that SIEP7 is not solvable (Xu 1998).

Theorem 4.4. (Xu 1998) Assume that all eigenvalues are distinct. Define

$$\Delta_k = \det \begin{bmatrix} 1 & \ldots & 1 & 1 & \ldots & 1 \\ \lambda_1 & \ldots & e_1^T \tilde{J} e_1 & \lambda_{k+1} & \ldots & \lambda_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{2n-1} & \ldots & e_1^T \tilde{J}^{2n-1} e_1 & \lambda_{k+1}^{2n-1} & \ldots & \lambda_{2n}^{2n-1} \end{bmatrix}.$$

Then SIEP9 has a unique solution if and only if

$$\Delta_k > 0$$

for all $k = 1, \ldots, 2n$.

A simple counterexample showing that SIEP9 is not always solvable is as follows. No 2×2 symmetric matrix J can have a fixed $(1, 1)$ entry a_1 and eigenvalues satisfying either $a_1 < \lambda_1 < \lambda_2$ or $\lambda_1 < \lambda_2 < a_1$.

4.4. Sensitivity issues

If the numerical computation is to be done using finite precision arithmetic, it is critical to understand the perturbation behaviour of the underlying mathematical problem. The notion of conditioning is normally used as an indication of the sensitivity dependence.

For IEPs, partly because inverse problems are, by nature, harder to analyse than direct problems, and partly because most IEPs have multiple solutions, not many results on sensitivity analysis have been performed. We believe that this is an important yet widely open area for further research.

We mentioned earlier that such a study could have the application of finding a robust solution that is least sensitive to perturbations. Despite the fact that considerable effort has been devoted to the development of numerical algorithms for many of the IEPs discussed in this article, we should make it clear that thus far very little attention has been paid to this direction. The analysis of either the conditioning of IEPs or the stability of the associated numerical methods is lacking. For that reason, we can only partially address the sensitivity issues by demonstrating known results for SIEP6a in this section. Clearly, more work needs to be done.

For the direct problem, it is easy to see that the function $F : \mathbb{R}^n \times \mathbb{R}_+^{n-1} \longrightarrow \mathbb{R}^{2n-1}$ where

$$F(a_1, \ldots, a_n, b_1, \ldots, b_{n-1}) = (\sigma(J), \sigma(\bar{J}))$$

is differentiable. The well-posedness of the inverse problem was initially established by Hochstadt (1974).

Theorem 4.5. (Hochstadt 1974) The unique solution J to SIEP6a depends continuously on the given data $\{\lambda_1, \ldots, \lambda_n\}$ and $\{\mu_1, \ldots, \mu_{n-1}\}$.

Mere continuous dependence is not enough for numerical computation. We need to quantify how the solution is perturbed by the change in problem data. Using the implicit function theorem, Hald (1976) refined this dependence and provided the following sensitivity dependence.

Theorem 4.6. (Hald 1976) Suppose J and $\tilde{J}$ are the solutions to SIEP6a with data

$$\lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{n-1} < \lambda_n,$$
$$\tilde{\lambda}_1 < \tilde{\mu}_1 < \tilde{\lambda}_2 < \cdots < \tilde{\mu}_{n-1} < \tilde{\lambda}_n,$$

respectively. Then there exists a constant K such that

$$\|J - \tilde{J}\|_F \le K \left(\sum_{i=1}^{n} |\lambda_i - \tilde{\lambda}_i|^2 + \sum_{i=1}^{n-1} |\mu_i - \tilde{\mu}_i|^2 \right)^{1/2}, \tag{4.7}$$

where the constant K depends on the quantities

$$d = \max\{\lambda_n, \tilde{\lambda}_n\} - \min\{\lambda_1, \tilde{\lambda}_1\},$$
$$\epsilon_0 = \frac{1}{d} \min_{j,k}\{|\lambda_j - \mu_k|, |\tilde{\lambda}_j - \tilde{\mu}_k|\},$$
$$\delta_0 = \frac{1}{2d} \min_{j \ne k}\{|\lambda_j - \lambda_k|, |\mu_j - \mu_k|, |\tilde{\lambda}_j - \tilde{\lambda}_k|, |\tilde{\mu}_j - \tilde{\mu}_k|, \}.$$

which measure the separation of the given data.

The constant K in (4.7) is significant in that it determines how the perturbation in the given data would be amplified. Its actual quantity, however, remains opaque because the implicit function theorem warrants only its existence but not its content. Xu (1993) introduced a form of condition number that could be explicitly estimated for SIEP6a. As a general rule, the smaller the separation of the given data, the more ill conditioned SIEP6a becomes.

4.5. Numerical methods

We mentioned earlier that numerical algorithms for a JIEP often followed directly from constructive proofs of its existence. Nevertheless, some of the procedures are subtly unstable. To save space, we shall not evaluate each method in this survey. Rather, we shall illustrate the basic ideas of two popular approaches: the Lanczos method (Parlett 1998) and the orthogonal reduction method (Boley and Golub 1987).

We first recall the following theorem, which is the basis of the Lanczos approach.

Theorem 4.7. The orthogonal matrix Q and the upper Hessenberg matrix H with positive subdiagonal entries can be completely determined by a given matrix A and the last (or any) column of Q if the relationship $Q^TAQ = H$ holds.

In our application, we want to construct the symmetric tridiagonal matrix $J = Q^T\Lambda Q$ with $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Thus, if the last column $\mathbf{q}_n$ is known, then the Jacobi matrix J can constructed in finitely many steps:

$$
\begin{aligned}
a_n &:= \mathbf{q}_n^T\Lambda\mathbf{q}_n,\\
b_{n-1} &:= \|\Lambda\mathbf{q}_n - a_n\mathbf{q}_n\|,\\
\mathbf{q}_{n-1} &:= (\Lambda\mathbf{q}_n - a_n\mathbf{q}_n)/b_{n-1},\\
&\text{for } i = 1, \ldots, n-2 \ \{\\
&\qquad a_{n-i} := \mathbf{q}_{n-i}^T\Lambda\mathbf{q}_{n-i},\\
&\qquad b_{n-i-1} := \|\Lambda\mathbf{q}_{n-i} - a_{n-i}\mathbf{q}_{n-i} - b_{n-i}\mathbf{q}_{n-i+1}\|,\\
&\qquad q_{n-i-1} := (\Lambda\mathbf{q}_{n-i} - a_{n-i}\mathbf{q}_{n-i} - b_{n-i}\mathbf{q}_{n-i+1})/b_{n-i-1},\\
&\qquad \}\\
a_1 &:= \mathbf{q}_1^T\Lambda\mathbf{q}_1.
\end{aligned}
$$

It only remains to calculate the column vector $\mathbf{q}_n$. To that end, we recall a classical result by Thompson and McEnteggert (1968).

Theorem 4.8. (Thompson and McEnteggert 1968) Let $(\lambda_i, \mathbf{x}_i)$, $i = 1, \ldots, n$, be orthonormal eigenpairs that form the spectral decomposition of

a given symmetric matrix A. Then

$$\text{adj}(\lambda_i I - A) = \prod_{\substack{k=1 \\ k\neq i}}^{n} (\lambda_i - \lambda_k)\mathbf{x}_i\mathbf{x}_i^T. \tag{4.8}$$

Evaluating both sides of (4.8) at the (n, n) position, we obtain

$$\det(\lambda_i I_{n-1} - A_{n-1}) = x_{ni}^2 \prod_{\substack{k=1 \\ k\neq i}}^{n} (\lambda_i - \lambda_k)$$

where x_{ni} is the last entry of $\mathbf{x}_i$, and recalling that A_{n-1} denotes the principal submatrix of size $n-1$. In our application, the last column $\mathbf{q}_n$ is precisely $[x_{n1}, \ldots, x_{nn}]^T$, if A is replaced by J. It follows that

$$x_{ni}^2 = \frac{\prod_{k=1}^{n-1}(\lambda_i - \mu_k)}{\prod_{\substack{k=1 \\ k\neq i}}^{n}(\lambda_i - \lambda_k)}. \tag{4.9}$$

In other words, the last column $\mathbf{q}_n$ for J can be expressed in terms of the spectral data $\{\lambda_1, \ldots, \lambda_n\}$ and $\{\mu_1, \ldots, \mu_{n-1}\}$. The Lanczos algorithm kicks in and SIEP6a is solved in finitely many steps.

We remark that other types of JIEP can be solved in similar ways with appropriate modifications, but we shall not examine them here. Readers are referred to the review paper by Boley and Golub (1987) and the book by Xu (1998) for more details.

We caution that the method by de Boor and Golub (1978) using the orthogonal polynomial approach is entirely equivalent to the above Lanczos approach, but is less stable in the face of roundoff error. We suggest that a reorthogonalization process should take place even along the Lanczos steps to ensure stability.

In the orthogonal reduction method, the given data are used first to construct a bordered diagonal matrix A of the form

$$A = \begin{bmatrix} \alpha & \beta_1 & \cdots & \beta_{n-1} \\ \beta_1 & \mu_1 & & 0 \\ \vdots & & \ddots & \\ \beta_{n-1} & 0 & \cdots & \mu_{n-1} \end{bmatrix}$$

so that $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$. Such a construction is entirely possible. First, α is trivially determined as $\alpha = \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n-1} \mu_i$. Secondly, the characteristic polynomial is given by

$$\det(\lambda I - A) = (\lambda - \alpha)\prod_{k=1}^{n-1}(\lambda - \mu_k) - \sum_{i=1}^{n-1} \beta_i^2 \left(\prod_{\substack{k=1 \\ k\neq i}}^{n-1} (\lambda - \mu_k) \right). \tag{4.10}$$

Thus, border elements β_i are given by

$$\beta_i^2 = -\frac{\prod_{k=1}^{n}(\mu_i - \lambda_k)}{\prod_{\substack{k=1 \\ k\neq i}}^{n-1}(\mu_i - \mu_k)}.$$

Let $\boldsymbol{\beta} = [\beta_1, \ldots, \beta_{n-1}]^T$. The next step is to construct an orthogonal matrix Q efficiently so that

$$\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & Q^T \end{bmatrix} A \begin{bmatrix} 1 & \mathbf{0}^T \\ 0 & Q \end{bmatrix} = \begin{bmatrix} \alpha & b_1\mathbf{e}_1^T \\ b_1\mathbf{e}_1 & \bar{J} \end{bmatrix} \tag{4.11}$$

becomes a Jacobi matrix. For this to happen, we must have $Q^T\boldsymbol{\beta} = b_1\mathbf{e}_1$, where $\mathbf{e}_1$ is the standard first coordinate vector in $\mathbb{R}^{n-1}$. It follows that $b_1 = \|\boldsymbol{\beta}\|$ and that the first column of Q is given by $\boldsymbol{\beta}/b_1$. The Lanczos procedure can now be employed to finish up the construction $Q^T\mathrm{diag}(\mu_1, \ldots, \mu_{n-1})Q = \bar{J}$ and SIEP6a is solved in finite steps.

Finally, we remark that other tridiagonalization process, including Householder transformations, Givens rotations, the Rutishauser method (Gragg and Harrod 1984), and so on, may also be used effectively to explore the bordered diagonal structure (Boley and Golub 1987).

5. Toeplitz inverse eigenvalue problems

Given a column vector $\mathbf{r} = [r_1, \ldots, r_n]^T$, a matrix $T = T(\mathbf{r})$ of the form

$$T := \begin{bmatrix} r_1 & r_2 & \ldots & r_{n-1} & r_n \\ r_2 & r_1 & & r_{n-2} & r_{n-1} \\ \vdots & \ddots & \ddots & & \vdots \\ r_{n-1} & & & r_1 & r_2 \\ r_n & r_{n-1} & & r_2 & r_1 \end{bmatrix}$$

is called a symmetric Toeplitz matrix. An inverse Toeplitz eigenvalue problem (**ToIEP**) concerns finding a vector $\mathbf{r} \in R^n$ so that $T(\mathbf{r})$ has a prescribed set of real numbers $\{\lambda_1, \ldots, \lambda_n\}$ as its spectrum. We mention in passing that a similar IEP could be asked for a Hankel matrix $H(\mathbf{r})$, related to the Toeplitz matrix $T(\mathbf{r})$ via $H(\mathbf{r}) = \Xi T(\mathbf{r})$ and $T(\mathbf{r}) = \Xi H(\mathbf{r})$, where Ξ is defined by (4.3). The set $\mathcal{T}(n)$ of symmetric Toeplitz matrices forms a subset of a larger class

$$\mathcal{C}(n) := \{M \in \mathbb{R}^{n\times n} | M = M^T, M = \Xi M \Xi\}$$

of centrosymmetric matrices, where Ξ is the unit perdiagonal matrix. A vector $\mathbf{v}$ is said to be *even* if $\Xi\mathbf{v} = \mathbf{v}$, and *odd* if $\Xi\mathbf{v} = -\mathbf{v}$. Every eigenspace of a centrosymmetric matrix, and hence a symmetric Toeplitz matrix, has a basis of even and odd vectors. The prescribed eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ in

a ToIEP, therefore, should also carry a corresponding parity assignment, as we shall explore in the next section.

5.1. Symmetry and parity

In Table 5.1 we summarize some characteristics of centrosymmetric matrices (Cantoni and Bulter 1976). Depending on whether n is even or odd, any centrosymmetric matrix M must assume the symmetry as is indicated in the second row of Table 5.1, where $A, C, \Xi \in \mathbb{R}^{\lfloor \frac{n}{2} \rfloor \times \lfloor \frac{n}{2} \rfloor}$, $\mathbf{x} \in \mathbb{R}^{\lfloor \frac{n}{2} \rfloor}$, $q \in \mathbb{R}$, and $A = A^T$. Let K be the orthogonal matrix defined in the table. Then M can be decomposed into 2×2 diagonal blocks via orthogonal similarity transformation by K. The blocks assume the forms shown in the last row of Table 5.1.

Table 5.1. Structure of centrosymmetric matrices

n	even	odd
M	$\begin{bmatrix} A & C^T \\ C & \Xi A \Xi \end{bmatrix}$	$\begin{bmatrix} A & x & C^T \\ x^T & q & x^T \Xi \\ C & \Xi x & \Xi A \Xi \end{bmatrix}$
$\sqrt{2}K$	$\begin{bmatrix} I & -\Xi \\ I & \Xi \end{bmatrix}$	$\begin{bmatrix} I & 0 & -\Xi \\ 0 & \sqrt{2} & 0 \\ I & 0 & \Xi \end{bmatrix}$
KMK^T	$\begin{bmatrix} A - \Xi C & 0 \\ 0 & A + \Xi C \end{bmatrix}$	$\begin{bmatrix} A - \Xi C & 0 & 0 \\ 0 & q & \sqrt{2} x^T \\ 0 & \sqrt{2} x & A + \Xi C \end{bmatrix}$

Effectively, the spectral decomposition of M is reduced to that of two submatrices of about half the size. If Z_1 denotes the $\lfloor \frac{n}{2} \rfloor \times \lfloor \frac{n}{2} \rfloor$ matrix of orthonormal eigenvectors for $A - \Xi C$, then columns from the matrix $K^T \begin{bmatrix} Z_1 \\ 0 \end{bmatrix}$ will be eigenvectors of M. These eigenvectors are odd vectors. Similarly, there are $\lceil \frac{n}{2} \rceil$ even eigenvectors of M computable from those of $A + \Xi C$ or

$$\begin{bmatrix} q & \sqrt{2} x^T \\ \sqrt{2} x & A + \Xi C \end{bmatrix}.$$

It is interesting to ask whether a symmetric Toeplitz matrix can have arbitrary spectrum with arbitrary parity. Can the parity be arbitrarily assigned to the prescribed eigenvalues in a ToIEP?

We consider the 3×3 case to further explore this question. Any matrix $M \in \mathcal{C}(3)$ is of the form

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ \times & m_{22} & \times \\ \times & \times & \times \end{bmatrix},$$

where quantities denoted by $\times$ can be obtained by symmetry. Without loss of generality, we may assume that the trace of M is zero. In this way, the parameters are reduced to m_{11}, m_{12} and m_{13}. Let $\mathcal{M}_{\mathcal{C}} = \mathcal{M}_{\mathcal{C}}(\lambda_1, \lambda_2, \lambda_3)$ denote the subset of centrosymmetric matrices with eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$. Assuming $\sum_{i=1}^{3} \lambda_i = 0$, elements in $\mathcal{M}_{\mathcal{C}}$ must satisfy the equations

$$\left(m_{11} - \frac{\lambda_{\varrho 1}}{4}\right)^2 + \frac{1}{2} m_{12}^2 = \frac{(\lambda_{\varrho 2} - \lambda_{\varrho 3})^2}{16},$$
$$m_{13} = m_{11} - \lambda_{\varrho 1},$$

where ϱ denotes any of the six permutations of $\{1, 2, 3\}$. Thus $\mathcal{M}_{\mathcal{C}}$ consists of three ellipses in $\mathbb{R}^3$. It suffices to plot these ellipses in the (m_{11}, m_{12})-plane only, since m_{13} is simply a shift of m_{11}. Several plots with qualitatively different eigenvalues are depicted in Figure 5.1. Observe that it is always the case that one circumscribes the other two.

A 3×3 centrosymmetric matrix is a solution to the ToIEP only if $m_{11} = 0$. By counting the number of m_{12}-intercepts, we should be able to know the number of solutions to the ToIEP. Specifically, we find that there are 4 solutions if eigenvalues are distinct and 2 solutions if one eigenvalue has multiplicity 2. We note further that the parities of the prescribed eigenvalues in the ToIEP cannot be arbitrary. Each of the ellipses corresponds to one particular parity assignment among the eigenvalues. An 'incorrect' parity assignment, such as the two smallest ellipses in the left column of Figure 5.1, implies that there is no m_{12}-intercept and, hence, no isospectral Toeplitz matrix. As far as the ToIEP is concerned, parity assignment is not explicitly given as part of the constraint. As a safeguard for ensuring existence, it has been suggested in the literature that the ordered eigenvalues should have alternating parity.

5.2. Existence

Despite the simplicity of the appearance of a ToIEP, the issue of its solvability has been quite a challenge. Delsarte and Genin (1984) argued that the problem would be analytically intractable if $n \geq 5$. Eventually, using a topological degree argument, Landau (1994) settled the following theorem with a nonconstructive proof.

Theorem 5.1. (Landau 1994) Every set of n real numbers is the spectrum of an $n \times n$ real symmetric Toeplitz matrix.

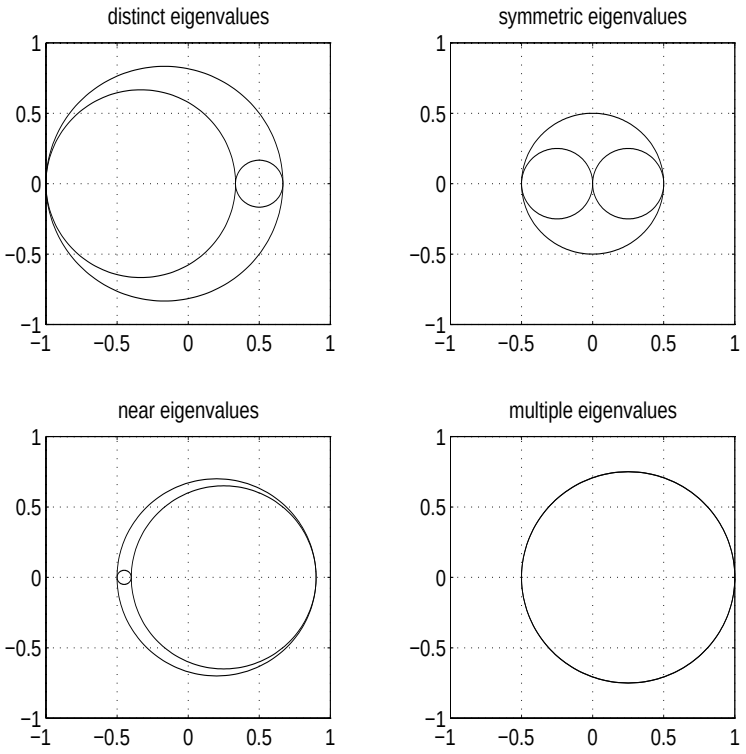

Figure 5.1. Plots of $\mathcal{M}_{\mathcal{C}}$ in the (m_{11}, m_{12})-plane

It might be worthwhile to briefly outline the proof as it shows the existence of a even more restricted class of Toeplitz matrices. A matrix $T(r_1, \ldots, r_n)$ is said to be *regular* if every principal submatrix $T(r_1, \ldots, r_k)$, $1 \le k \le n$, has distinct eigenvalues that, when arranged in ascending order, alternate parity with the largest one even. Consider the map $\varphi : \mathbb{R}^{n-2} \longrightarrow \mathbb{R}^{n-2}$ defined by

$$\varphi(t_3, \ldots, t_n) = (y_2, \ldots, y_{n-1}),$$

where $y_i := -\frac{\mu_i}{\mu_1}$, $i = 2, \ldots, n-1$, if $\mu_1 \le \cdots \le \mu_n$ are eigenvalues of $T(0, 1, t_3, \ldots, t_n)$. Note that since $\sum_{i=1}^{n} \mu_i = 0$, it is necessary that $\mu_1 < 0$. The range of φ is the simplex

$$\Delta := \left\{ (y_2, \ldots, y_{n-1}) \in \mathbb{R}^{n-2} \middle| \begin{array}{c} -1 \le y_2 \le \cdots \le y_{n-1} \\ y_2 + \cdots + y_{n-2} + 2y_{n-1} \le 1 \end{array} \right\}.$$

The key components in the proof by Landau (1994) are as follows. First, the set $\mathcal{F}$ of regular Toeplitz matrices of the form $T(0, 1, t_3, \ldots, t_n)$ is not empty.

Secondly, the map φ restricted to those points $(t_3, \ldots, t_n) \in \mathbb{R}^{n-2}$ such that $T(0, 1, t_3, \ldots, t_n) \in \mathcal{F}$ is a *surjective* map onto the interior of Δ. Finally, any given $\lambda_1 \leq \cdots \leq \lambda_n$ can be *shifted* and *scaled* to a unique point in Δ whose pre-image(s) can then be scaled and shifted backward to a symmetric Toeplitz matrix with eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$.

5.3. Numerical methods

Lack of understanding does not necessarily preclude the development of effective numerical algorithms for the ToIEP. There are two basic approaches to tackle the ToIEP numerically: one by iteration and the other by continuation. We briefly describe the basic ideas for each approach in this section.

Regarding the ToIEP as a nonlinear system of n equations in n unknowns, a natural tactic would be a Newton-type iteration. The schemes in Friedland, Nocedal and Overton (1987), originally proposed for the more general class of PIEPs, are in this class and certainly applicable to the ToIEP. The inverse Rayleigh quotient algorithm in Laurie (1988, 1991) is also equivalent to a Newton-type variation. These methods do not exploit the Toeplitz structure and can suffer from local convergence. The iterative scheme proposed by Trench (1997) seems to have more robust performance, but still no global convergence can be proved. The following discussion is another Newton-type iteration by Chu (1994). The iterations are confined in $\mathcal{C}(n)$. Since the centrosymmetric structure is preserved, the cost is substantially reduced and the case of double eigenvalues can be handled effectively.

Recall that the classical Newton method

$$x^{(\nu+1)} = x^{(\nu)} - (f'(x^{(\nu)}))^{-1} f(x^{(\nu)})$$

for a scalar function $f : \mathbb{R} \longrightarrow \mathbb{R}$ can be thought of as two steps: the tangent step where $x^{(\nu+1)}$ is the x-intercept of the tangent line from $(x^{(\nu)}, f(x^{(\nu)}))$ of the graph of f, and the lift step where the point $(x^{(\nu+1)}, f(x^{(\nu+1)}))$ is a natural lift of the intercept along the y-axis to the graph of f. Let the given eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ be arranged as $\boldsymbol{\lambda} = [\phi_1, \ldots, \phi_{\lfloor \frac{n}{2} \rfloor}, \psi_1, \ldots, \psi_{\lceil \frac{n}{2} \rceil}]^T$, where ϕ_k and ψ_k are of parity even and odd, respectively. Let $\Lambda = \mathrm{diag}(\boldsymbol{\lambda})$. An analogue of this idea applied to the ToIEP is to think of the isospectral subset $\mathcal{M}_{\mathcal{C}} = \mathcal{M}_{\mathcal{C}}(\Lambda)$ of centrosymmetric matrices as the graph of some unknown f, and the subspace $\mathcal{T}(n)$ of symmetric Toeplitz matrices as the x-axis. We want to do the tangent and lift iterations between these two entities. From Section 5.1, we see that every element $M \in \mathcal{M}_{\mathcal{C}}$ can be characterized by the parameter $Z \in \mathcal{O}(\lfloor \frac{n}{2} \rfloor) \times \mathcal{O}(\lceil \frac{n}{2} \rceil)$ where $M = Q\Lambda Q^T$ and $Z = KQ$. It follows that tangent vectors of $\mathcal{M}_{\mathcal{C}}$ at M are of the form

$$T_M(\mathcal{M}_{\mathcal{C}}) = \tilde{S}M - M\tilde{S}, \tag{5.1}$$

with $\tilde{S} := Q\mathrm{diag}(S_1, S_2)Q^T$ where S_1 and S_2 are arbitrary skew-symmetric

matrices in $\mathbb{R}^{\lfloor \frac{n}{2} \rfloor \times \lfloor \frac{n}{2} \rfloor}$ and $\mathbb{R}^{\lceil \frac{n}{2} \rceil \times \lceil \frac{n}{2} \rceil}$, respectively. Thus a tangent step from a given $M^{(\nu)} \in \mathcal{M}_{\mathcal{C}}(\Lambda)$ amounts to finding a skew-symmetric matrix $\tilde{S}^{(\nu)}$ and a vector $\mathbf{r}^{(\nu+1)}$ so that

$$M^{(\nu)} + \tilde{S}^{(\nu)} M^{(\nu)} - M^{(\nu)} \tilde{S}^{(\nu)} = T(\mathbf{r}^{(\nu+1)}). \tag{5.2}$$

Assume the spectral decomposition $M^{(\nu)} = Q^{(\nu)} \Lambda {Q^{(\nu)}}^T$ and define $Z^{(\nu)} = KQ^{(\nu)}$. The tangent equation is reduced to

$$\Lambda + S^{(\nu)} \Lambda - \Lambda S^{(\nu)} = {Z^{(\nu)}}^T \big(KT(\mathbf{r}^{(\nu+1)}) K^T \big) Z^{(\nu)}, \tag{5.3}$$

where $S^{(\nu)} = {Q^{(\nu)}}^T \tilde{S}^{(\nu)} Q^{(\nu)}$ remains skew-symmetric. Note that the product $KT(\mathbf{r}^{(\nu+1)})K^T$ is a 2×2 diagonal block matrix, denoted by $\mathrm{diag}(T_1^{(\nu+1)}, T_2^{(\nu+1)})$, because $T(\mathbf{r}^{(\nu+1)})$ is centrosymmetric. We also know from the discussion in Section 5.1 that $Z^{(\nu)} = \mathrm{diag}(Z_1^{(\nu)}, Z_2^{(\nu)})$. Thus the system (5.3) is effectively split in half.

We first retrieve the vector $\mathbf{r}^{(\nu+1)}$. It suffices to compare the diagonal elements on both sides without reference to $S^{(\nu)}$. Note that the right-hand side of (5.3) is linear in $\mathbf{r}^{(\nu+1)}$. This linear relationship can be expressed as

$$\Omega^{(\nu)} \mathbf{r}^{(\nu+1)} = \lambda$$

for $r^{(\nu+1)}$, where the entries in the matrix $\Omega^{(\nu)} = [\Omega_{ij}^{(\nu)}]$ are defined by

$$\Omega_{ij}^{(\nu)} := \begin{cases} (Z_1^{(\nu)})_{*i}^T E_1^{[j]} (Z_1^{(\nu)})_{*i}, & \text{if } 1 \le i \le \lfloor \frac{n}{2} \rfloor, \\ (Z_2^{(\nu)})_{*i}^T E_2^{[j]} (Z_2^{(\nu)})_{*i}, & \text{if } \lfloor \frac{n}{2} \rfloor < i \le n. \end{cases}$$

In the above, $E_1^{[j]}$ and $E_2^{[j]}$ are the diagonal blocks in the 2×2 diagonal block matrix $KT(\mathbf{e}_j)K^T$, $\mathbf{e}_j$ is the jth standard unit vector, and $(Z_k^{(\nu)})_{*i}$ denotes the ith column of the matrix $Z_k^{(\nu)}$. Throughout the calculations, we need only to multiply vectors or matrices of lengths $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$. Once $T(\mathbf{r}^{(\nu+1)})$ is determined, off-diagonal elements in (5.3) determine $S^{(\nu)}$. Specifically, if eigenvalues within each parity group are distinct, then it is easy to see that

$$(S_1^{(\nu)})_{ij} = \frac{(Z_1^{(\nu)})_{*i}^T T_1^{(\nu+1)} (Z_1^{(\nu)})_{*j}}{\phi_i - \phi_j}, \quad 1 \le i < j \le \left\lfloor \frac{n}{2} \right\rfloor,$$

$$(S_2^{(\nu)})_{ij} = \frac{(Z_2^{(\nu)})_{*i}^T T_2^{(\nu+1)} (Z_2^{(\nu)})_{*j}}{\psi_i - \psi_j}, \quad 1 \le i < j \le \left\lceil \frac{n}{2} \right\rceil.$$

This completes the calculation for the tangent step. We remark that the scheme is capable of handling the case of double eigenvalues because such eigenvalues have to be split into one even and one odd (Delsarte and Genin 1984).

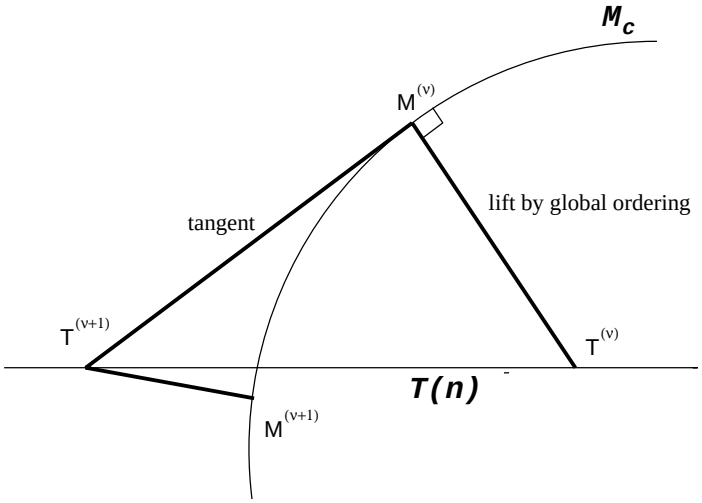

Figure 5.2. Geometry of lift by Wielandt–Hoffman theorem

To obtain a lift from $T(r^{(\nu+1)})$ to $\mathcal{M}_{\mathcal{C}}$, we look for any matrix $M^{(\nu+1)} \in \mathcal{M}_{\mathcal{C}}$ that is nearest to $T(r^{(\nu+1)})$. The idea is sketched in Figure 5.2. Such a nearest approximation can be obtained by the Wielandt–Hoffman theorem. That is, suppose the spectral decomposition of $T(r^{(\nu+1)})$ is given by

$$\overline{Z}^{(\nu+1)^T} K T(r^{(\nu+1)}) K^T \overline{Z}^{(\nu+1)} = \begin{bmatrix} \overline{\Lambda}_1^{(\nu+1)} & 0 \\ 0 & \overline{\Lambda}_2^{(\nu+1)} \end{bmatrix}.$$

Rearrange $\{\lambda_1, \ldots, \lambda_n\}$ in the same ordering as in $\overline{\Lambda}_1^{(\nu+1)}$ and $\overline{\Lambda}_2^{(\nu+1)}$ to obtain $\tilde{\Lambda}_1^{(\nu+1)}$ and $\tilde{\Lambda}_2^{(\nu+1)}$. Then the nearest approximation $M^{(\nu+1)}$ is given by

$$M^{(\nu+1)} := K^T \overline{Z}^{(\nu+1)} \mathrm{diag}\big(\tilde{\Lambda}_1^{(\nu+1)}, \tilde{\Lambda}_2^{(\nu+1)}\big) \overline{Z}^{(\nu+1)^T} K.$$

For computational purposes, this $M^{(\nu+1)}$ never needs to be calculated. We only need to repeat the tangent step with the new parity assignment $\Lambda = \mathrm{diag}(\tilde{\Lambda}_1^{(\nu+1)}, \tilde{\Lambda}_2^{(\nu+1)})$ and the new parameter matrix $Z^{(\nu+1)} := \overline{Z}^{(\nu+1)}$. It can be proved that this method converges quadratically.

An alternative approach, fundamentally different from iterative methods, is to solve the ToIEP by tracing curves defined by differential equations. The idea is to continually transform the matrix Λ until a Toeplitz matrix is found. One such formulation is the initial value problem

$$\begin{cases} \frac{\mathrm{d}X}{\mathrm{d}t} &= [X, k(X)], \\ X(0) &= \Lambda, \end{cases} \tag{5.4}$$

where $[A, B] := AB - BA$ denotes the Lie bracket and $k(X) = [k_{ij}(X)]$ is defined by

$$k_{ij}(X) := \begin{cases} x_{i+1,j} - x_{i,j-1}, & \text{if } 1 \le i < j \le n, \\ 0, & \text{if } 1 \le i = j \le n, \\ x_{i,j-1} - x_{i+1,j}, & \text{if } 1 \le j < i \le n. \end{cases} \tag{5.5}$$

The skew-symmetry of $k(X)$ guarantees that the solution $X(t)$ to (5.4) exists for all t and enjoys a continuous spectral decomposition $X(t) = Q(t)^T \Lambda Q(t)$. The orthogonal transformation $Q(t)$ is determined by the initial value problem

$$\begin{cases} \frac{\mathrm{d}Q}{\mathrm{d}t} &= Qk(Q^T \Lambda Q), \\ Q(0) &= I. \end{cases} \tag{5.6}$$

The limiting behaviour of $Q(t)$ determines the limiting behaviour of $X(t)$ and *vice versa*. The hope of getting a solution to the ToIEP hinges upon that $k(X) = 0$ if and only if X is a Toeplitz matrix. For that reason, $k(X)$ is called a Toeplitz annihilator in Chu (1993). The system (5.6) can be integrated effectively by available geometric integrators (Calvo, Iserles and Zanna 1997, Dieci, Russel and Vleck 1994, Iserles, Munthe-Kaas, Nørsett and Zanna 2000). Numerical experiences seem to suggest that the flows always converge, but a rigorous proof is still missing.

6. Nonnegative inverse eigenvalue problems

The nonnegative inverse eigenvalue problem (**NIEP**) concerns the construction of an entry-wise nonnegative matrix $A \in \mathbb{R}^{n \times n}$ with a prescribed set $\{\lambda_1, \ldots, \lambda_n\}$ of eigenvalues. Partially because of the important Perron–Frobenius theory, this inverse problem has drawn considerable interest in the literature.

The earliest study of this subject was perhaps that of the Russian mathematician Suleĭmanova (1949) on stochastic matrices, followed by Perfect (1953, 1955). The first systematic treatment of eigenvalues of symmetric nonnegative matrices should probably be attributed to Fiedler (1974). A more comprehensive study was conducted by Boyle and Handelman (1991), using the notion of symbolic dynamics to characterize the conditions that a given set is a portion of the spectrum of a nonnegative matrix or primitive matrix. General treatises on nonnegative matrices and their applications include the books by Berman and Plemmons (1979) and Minc (1988). Both books devote extensive discussion to the NIEP as well. Most of the discussions in the literature centre around finding conditions to qualify a given set of values as the spectrum of some nonnegative matrices. A short list of references giving various necessary or sufficient conditions includes Barrett

and Johnson (1984), Boyle and Handelman (1991), Friedland (1978), Friedland and Melkman (1979), Loewy and London (1978), de Oliveira (1983) and Reams (1996). The trouble is that the necessary condition is usually too general and the sufficient condition too specific. Under a few special sufficient conditions, the nonnegative matrices can be constructed numerically (Soules 1983). General numerical treatments for NIEPs, even knowing the existence of a solution, are not available at the time of writing.

A further refinement in the posing of NIEPs has also attracted some attention. Suppose the given eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ are all real. The real-valued nonnegative inverse eigenvalue problem (**RNIEP**) concerns which set of values $\{\lambda_1, \ldots, \lambda_n\}$ occurs as the spectrum of a nonnegative matrix. The symmetric nonnegative inverse eigenvalue problem (**SNIEP**) concerns which set occurs as the spectrum of a symmetric nonnegative matrix. It was proved that there exist real numbers $\{\lambda_1, \ldots, \lambda_n\}$ that solve the RNIEP but *not* the SNIEP (Guo 1996, Johnson, Laffey and Loewy 1996).

6.1. Some existence results

The solvability of the NIEP has been the major issue of discussion in the literature. Existence results, either necessary or sufficient, are too numerous to be listed here. We shall mention only two results that, in some sense, provide the most distinct criteria in this regard.

Given a matrix A, the moments of A are defined to be the sequence of numbers $s_k = \mathrm{trace}(A^k)$. Recall that, if $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$, then

$$s_k = \sum_{i=1}^{n} \lambda_i^k.$$

For nonnegative matrices, the moments are always nonnegative. The following necessary condition is due to Loewy and London (1978).

Theorem 6.1. (Loewy and London 1978) Suppose $\{\lambda_1, \ldots, \lambda_n\}$ are eigenvalues of an $n \times n$ nonnegative matrix. Then the inequalities

$$s_k^m \leq n^{m-1} s_{km} \tag{6.1}$$

hold for all $k, m = 1, 2, \ldots$.

Note also that the inequalities in (6.1) are sharp, being equalities for the identity matrix. If we further limit the inverse problem to positive matrices, that is, every entry exceeds zero, it turns out that the eigenvalues can be completely characterized. The following necessary and sufficient condition appeared at the end of the long treatise by Boyle and Handelman (1991, p. 313)

Theorem 6.2. (Boyle and Handelman 1991) The set $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$, with $\lambda_1 = \max_{1 \le i \le n} |\lambda_i|$, is the nonzero spectrum of a positive matrix of size $m \ge n$ if and only if:

(1) $\lambda_1 > |\lambda_i|$ for all $i > 1$,
(2) $s_k > 0$ for all $k = 1, 2, \ldots,$ and
(3) $\prod_{i=1}^{n}(t - \lambda_i)$ has real coefficients in t.

6.2. Symmetric nonnegative inverse eigenvalue problem

We shall discuss a least squares approach for solving the general NIEP in the next section. At present, we touch upon the SNIEP with a few more comments.

First, we remark that solvability of SNIEPs remains open. Some sufficient conditions are listed in Berman and Plemmons (1979, Chapter 4). The set $\lambda = \{\sqrt[3]{51} + \epsilon, 1, 1, 1, -3, -3\}$, with $\epsilon > 0$, however, does not satisfy any of these conditions. In fact, it cannot be the nonzero spectrum of any symmetric nonnegative matrix (Johnson *et al.* 1996).

Friedland and Melkman (1979) limited the consideration of NIEPs to symmetric tridiagonal structure. A simple result can be established.

Theorem 6.3. (Friedland and Melkman 1979) A set of real numbers $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ is the spectrum of an $n \times n$ nonnegative tridiagonal matrix if and only if $\lambda_i + \lambda_{n-i+1} = 0$ for all i. In this case, the matrix is given by $J = \mathrm{diag}(A_1, \cdots, A_{[(n+1)/2]})$, where

$$A_i = \frac{1}{2}\begin{bmatrix} \lambda_i + \lambda_{n-i+1} & \lambda_i - \lambda_{n-i+1} \\ \lambda_i - \lambda_{n-i+1} & \lambda_i + \lambda_{n-i+1} \end{bmatrix}, \quad 1 \le i < (n+1)/2,$$

and $A_i = [\lambda_i]$, if $i = (n+1)/2$ and n is odd.

Note that the matrix J constructed above is reducible when $n > 2$. The problem becomes harder if J is required further to be Jacobi, that is, with positive subdiagonal elements. One sufficient but not necessary condition for this particular JIEP is as follows.

Theorem 6.4. If $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ and if $\lambda_i + \lambda_{n-i+1} > 0$ for all i, then there exists a positive Jacobi matrix with spectrum $\{\lambda_1, \ldots, \lambda_n\}$.

It would be very difficult to achieve a nearly simple characterization of solution to a general SNIEP. On the other hand, we could formulate the SNIEP as a constrained optimization problem of *minimizing* the objective function

$$F(Q, R) := \frac{1}{2}\|Q^T \Lambda Q - R \circ R\|^2,$$

subject to the constraint $(Q, R) \in \mathcal{O}(n) \times \mathcal{S}(n)$, where $\circ$ stands for the

Hadamard product and $\mathcal{S}(n)$ stands for the subspace of $n \times n$ symmetric matrices. The idea is to parametrize any symmetric matrix $X = Q^T \Lambda Q$ that is isospectral to Λ by the orthogonal matrix Q and to parametrize any symmetric nonnegative matrix $Y = R \circ R$ by the symmetric matrix R via entry-wise squares. The SNIEP is solvable if and only if $F(Q, R) = 0$ for some Q and R. Such a formulation offers a handle for numerical computation by optimization techniques. In Chu and Driessel (1991), the dynamical system

$$\begin{cases} \frac{\mathrm{d}X}{\mathrm{d}t} &= [X, [X, Y]], \\ \frac{\mathrm{d}Y}{\mathrm{d}t} &= 4Y \circ (X - Y), \end{cases} \tag{6.2}$$

resulting from projected gradient flow, was studied as a possible numerical means for solving the SNIEP. It is interesting to note that, even if the SNIEP is not solvable, the limit point of the gradient flow gives rise to a least squares solution. We shall discuss an analogous approach of (6.2) for NIEPs in Section 7.2.

7. Stochastic inverse eigenvalue problems

An $n \times n$ nonnegative matrix is a (row) stochastic matrix if all its row sums are 1. The stochastic inverse eigenvalue problem (**StIEP**) concerns the construction of a stochastic matrix with prescribed spectrum. Clearly the StIEP is a special NIEP with the additional row sum structure. It should be noted that, in contrast to the linearly constrained IEPs discussed thus far, the structure involved in the StIEP is nonlinear in the sense that the sum of two (stochastic) structured matrices does not have the same (stochastic) structure.

The Perron–Frobenius theorem asserts that the spectral radius $\rho(A)$ of an irreducible nonnegative matrix A is a positive maximal eigenvalue of A. The corresponding maximal eigenvector can be chosen to have all elements positive. Recall also that the set of reducible matrices forms a subset of measure zero. With this in mind, the spectral properties for stochastic matrices do not differ much from those of other nonnegative matrices, because of the following result (Minc 1988).

Theorem 7.1. Let A be any nonnegative matrix with positive maximal eigenvalue r and a positive maximal eigenvector x. Let $D = \mathrm{diag}(x)$. Then $D^{-1} r^{-1} A D$ is a stochastic matrix.

This motivates the notion that, if we could construct a generic solution for the NIEP, then we would also solve the StIEP by a diagonal similarity transformation. We shall pursue this idea as a numerical method for both the NIEP and the StIEP.

7.1. Existence

The StIEP is a hard problem. First, we note that Minc (1988) called our StIEP an inverse *spectrum* problem to distinguish it carefully from the problem of determining conditions under which one single complex number is an eigenvalue of a stochastic matrix. For the latter, the set Θ_n of points in the complex plane that are eigenvalues of any $n \times n$ stochastic matrices has been completely characterized by Karpelevič (1951). The complete statement of Karpelevič's theorem is rather lengthy (Minc 1988, Theorem 1.8), so we shall only highlight the main points below.

Theorem 7.2. (Karpelevič 1951) The region Θ_n is contained in the unit disk and is symmetric with respect to the real axis. It intersects the unit circle at points $e^{2\pi ia/b}$ where a and b range over all integers such that $0 \le a < b \le n$. The boundary of Θ_n consists of curvilinear arcs connecting these points in circular order. Any λ on these arcs must satisfy one of these equations:

$$\lambda^q(\lambda^p - t)^r = (1-t)^r,$$
$$(\lambda^b - t)^d = (1-t)^d\lambda^q,$$

where $0 \le t \le 1$, and b, d, p, q, r are natural integers determined by certain specific rules (explicitly given in Karpelevič (1951) and Minc (1988)).

An example Θ_4 is sketched in Figure 7.1. It should be stressed that the Karpelevič theorem characterizes only one complex value a time. It does not provide further insights into when two or more points in Θ_n are eigenvalues of the *same* stochastic matrix. It provides only a necessary condition for the StIEP.

We conclude this section with perhaps the first sufficient condition due to Suleĭmanova (1949).

Theorem 7.3. (Suleĭmanova 1949) The n real numbers $1, \lambda_1, \ldots, \lambda_{n-1}$, with $|\lambda_j| < 1$, are the characteristic values of some positive stochastic matrix of order n if $\sum |\lambda_j| < 1$, where the sum is over the js with $\lambda_j < 0$. If the λ_js are all negative the condition is also necessary.

Most of the sufficient conditions for the StIEP are imposed on real eigenvalues (Suleĭmanova 1949, Perfect 1953, Perfect 1955). If the resulting nonnegative matrix is generic, then the sufficient conditions for the NIEP also apply to the StIEP by Theorem 7.1.

7.2. Numerical method

It appears that, except for Soules (1983), none of the proofs for sufficient conditions is constructive, and no numerical algorithms are available even if a sufficient condition is satisfied. Even with Soules (1983), the construction

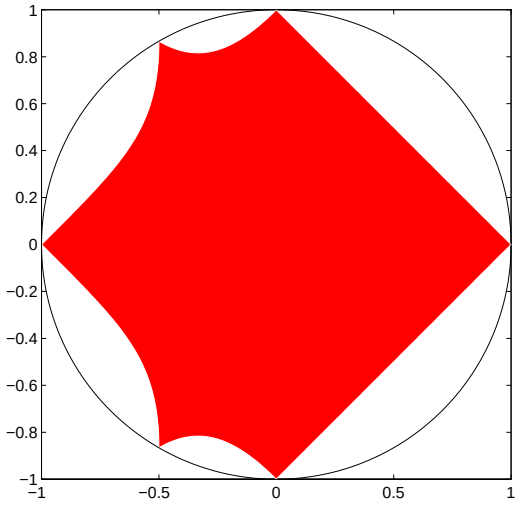

Figure 7.1. Θ_4 by the Karpelevič theorem

is limited in that the components of the Perron eigenvector need to satisfy additional inequalities. Recently, Chu and Guo (1998) proposed the following least squares approach that might be employed to solve the NIEP and the StIEP for generally prescribed eigenvalues. The idea is parallel to that of (6.2).

The diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ can be transformed, if necessary, into a diagonal block matrix with 2×2 real blocks, if some of the given eigenvalues appear in complex conjugate pairs. The set of isospectral matrices

$$\mathcal{M}(\Lambda) = \{P\Lambda P^{-1} \in \mathbb{R}^{n\times n} | P \in \mathbb{R}^{n\times n} \text{ is nonsingular}\}$$

is parametrized by nonsingular matrices P. The cone of nonnegative matrices

$$\pi(R^n_+) := \{B \circ B | B \in \mathbb{R}^{n\times n}\}$$

is characterized by the Hadamard product of general square matrices. A solution to the NIEP must be at the intersection of $\mathcal{M}(\Lambda)$ and $\pi(R^n+)$, if there is any. If such a nonnegative matrix has positive maximal eigenvector, it can be reduced to a stochastic matrix by diagonal similarity transformation. We thus formulate the constrained optimization problem

$$\begin{aligned} \text{Minimize} \quad & F(P,R) := \frac{1}{2}\|P\Lambda P^{-1} - R \circ R\|^2, \\ \text{Subject to} \quad & P \in Gl(n),\ R \in \mathbb{R}^{n\times n}, \end{aligned}$$

where $Gl(n)$ denotes the group of invertible matrices in $\mathbb{R}^{n\times n}$. We use P and R as variables to manoeuvre elements in $\mathcal{M}(\Lambda)$ and $\pi(R^n_+)$ to reduce

the objective value. Clearly, the feasible sets are open and a minimum may not exist. With respect to the induced inner product,

$$\langle (X_1, Y_1), (X_2, Y_2) \rangle := \langle X_1, X_2 \rangle + \langle Y_1, Y_2 \rangle,$$

in the product topology of $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$, the gradient of F can be expressed as the pair

$$\nabla F(P, R) = ([\Delta(P, R), M(P)^T]P^{-T}, -2\Delta(P, R) \circ R), \tag{7.1}$$

where we recall that $[\cdot, \cdot]$ is the Lie bracket, and we abbreviate

$$M(P) := P\Lambda P^{-1},$$
$$\Delta(P, R) := M(P) - R \circ R.$$

The differential system

$$\frac{\mathrm{d}P}{\mathrm{d}t} = [M(P)^T, \Delta(P, R)]P^{-T},$$
$$\frac{\mathrm{d}R}{\mathrm{d}t} = 2\Delta(P, R) \circ R$$

thus provides a steepest descent flow on the feasible set $Gl(n) \times \mathbb{R}^{n\times n}$ for the objective function $F(P, R)$.

There is an unexpected advantage that deserves mentioning. Note that the zero structure in the original matrix $R(0)$ is preserved throughout the integration due to the Hadamard product. This feature may be exploited to construct a Markov chain with prescribed linkages and spectrum. That is, if it is desirable that state i is not allowed to transit into state j, we assign $r_{ij} = 0$ in the initial value $R(0)$. That zero transit status is maintained throughout the evolution.

On the other hand, the solution flow $P(t)$ is susceptible to becoming singular and the involvement of P^{-1} is somewhat worrisome. A remedy is to monitor the analytic singular value decomposition (ASVD) (Bunse-Gerstner, Byers, Mehrmann and Nichols 1991, Wright 1992),

$$P(t) = X(t)S(t)Y(t)^T, \tag{7.2}$$

of the path of matrices $P(t)$. In (7.2), $X(t)$ and $Y(t)$ are orthogonal, $S(t)$ is diagonal, and all are analytic in t. Such an ASVD flow exists because the solution $P(t)$ is analytic, by the Cauchy–Kovalevskaya theorem. The flow $(P(t), R(t))$ is equivalent to the flow of $(X(t), S(t), Y(t), R(t))$, where the differential equations governing $X(t), S(t)$, and $Y(t)$ can be obtained as follows (Wright 1992).

On differentiating (7.2), we have

$$X^T \frac{\mathrm{d}P}{\mathrm{d}t} Y = \underbrace{X^T \frac{\mathrm{d}X}{\mathrm{d}t}}_{Z} S + \frac{\mathrm{d}S}{\mathrm{d}t} + S \underbrace{\frac{\mathrm{d}Y^T}{\mathrm{d}t} Y}_{W}, \tag{7.3}$$

where both Z and W are skew-symmetric matrices. Note that $Q := X^T \frac{\mathrm{d}P}{\mathrm{d}t} Y$ is known since $\frac{\mathrm{d}P}{\mathrm{d}t}$ is already specified. Note also that the inverse of $P(t)$ can be calculated from $P^{-1} = YS^{-1}X^T$, whereas the diagonal entries of $S = \mathrm{diag}\{s_1, \ldots, s_n\}$ provide us with information about the proximity of $P(t)$ to singularity. The diagonals on both sides of (7.3) lead to the equation

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \mathrm{diag}(Q)$$

for $S(t)$. The off-diagonals on both sides of (7.3) give rise to

$$\frac{\mathrm{d}X}{\mathrm{d}t} = XZ,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = YW,$$

where, if $s_k^2 \neq s_j^2$ (in case of equality, W and Z can still be obtained by other means (Wright 1992)), entries of W and Z are obtained from

$$z_{jk} = \frac{s_k q_{jk} + s_j q_{kj}}{s_k^2 - s_j^2},$$
$$w_{jk} = \frac{s_j q_{jk} + s_k q_{kj}}{s_j^2 - s_k^2}$$

for all $j > k$. The flow is now ready to be integrated by available geometric integrators.

We note that $\frac{\mathrm{d}F(P(t),R(t))}{\mathrm{d}t} = -\|\nabla F(P(t), R(t))\|^2 \leq 0$. Thus the method fails to solve the NIEP only in two situations: either $P(t)$ becomes singular in finite time or $F(P(t), R(t))$ converges to a least squares local solution. In the former case, a restart might avoid the problem. In the latter case, either the NIEP has no solution at all or the algorithm needs a restart.

8. Unitary Hessenberg inverse eigenvalue problems

Eigenvalues of unitary matrices are on the unit circle. The unitary Hessenberg inverse eigenvalue problem (**UHIEP**) concerns the construction of an unitary Hessenberg matrix with prescribed points on the unit circle. Eigenvalue problems for unitary Hessenberg matrices arise naturally in several signal processing applications, including the frequency estimation procedure and the harmonic retrieval problem for radar or sonar navigation. The characteristic polynomials of unitary Hessenberg matrices are the well-known Szegő polynomials. The Szegő polynomials are orthogonal with respect to a certain measure on the unit circle in just the same way as the characteristic polynomials of the Jacobi matrices are orthogonal with respect to a certain weight on an interval. In many ways, theory and algorithms for the UHIEP are similar to those for the JIEP. Most of the discussion in this section are

results adapted from Ammar, Gragg and Reichel (1991) and Ammar and He (1995).

Any upper Hessenberg unitary matrix H with positive subdiagonal entries can uniquely expressed as the product

$$H = G_1(\eta_1) \cdots G_{n-1}(\eta_{n-1}) \tilde{G}_n(\eta_n), \tag{8.1}$$

where η_k are complex numbers with $|\eta_k| < 1$ for $1 \le k < n$ and $|\eta_n| = 1$, each $G_k(\eta_k)$, $k = 1, \ldots, n-1$ is a Givens rotation,

$$G_k(\eta_k) = \begin{bmatrix} I_{k-1} & & & \\ & -\eta_k & \zeta_k & \\ & \zeta_k & \bar{\eta}_k & \\ & & & I_{n-k+1} \end{bmatrix}$$

with $\zeta_k := \sqrt{1 - |\eta_k|^2}$ and $\tilde{G}_n(\eta_n) = \text{diag}[I_{n-1}, -\eta_n]$. In other words, each unitary upper Hessenberg matrix is determined by $2n-1$ *real* parameters, another feature analogous to the Jacobi matrices. For convenience, this dependence is denoted by $H = H(\eta_1, \ldots, \eta_n)$. The decomposition (8.1), referred to as the Schur parametrization, plays a fundamental role in efficient algorithms for upper Hessenberg unitary matrices.

There are considerable similarities between UHIEPs and JIEPs. The following two UHIEPs, for example, are analogous to SIEP8 and SIEP6a, respectively.

Theorem 8.1. (Ammar and He 1995) Given two sets $\{\lambda_1, \ldots, \lambda_n\}$ and $\{\mu_1, \ldots, \mu_n\}$ of strictly interlaced points on the unit circle, there exist a unique unitary upper Hessenberg matrix $H = H(\eta_1, \ldots, \eta_n)$ and a unique complex number α of unit modulus such that $\sigma(H) = \{\lambda_1, \ldots, \lambda_n\}$ and $\sigma(H(\alpha\eta_1, \ldots, \alpha\eta_n)) = \{\mu_1, \ldots, \mu_n\}$.

Note that the matrix $\tilde{H} = H(\alpha\eta_1, \ldots, \alpha\eta_n)$ is a rank-one perturbation of $H(\eta_1, \ldots, \eta_n)$, because $\tilde{H} = (I - (1-\alpha)\mathbf{e}_1\mathbf{e}_1^T)H$. Unlike SIEP8, this perturbation does not just affect the (n, n) entry.

The leading principal submatrix H_{n-1} of a unitary matrix is not unitary, and its eigenvalues do not lie on the unit circle. One way to modify the notion of submatrix is as follows.

Theorem 8.2. (Ammar and He 1995) Given two sets of strictly interlaced points $\{\lambda_1, \ldots, \lambda_n\}$ and $\{\mu_0, \mu_1, \ldots, \mu_{n-1}\}$ on the unit circle, there exist a unique unitary upper Hessenberg matrix $H = H(\eta_1, \ldots, \eta_n)$ such that $\sigma(H) = \{\lambda_1, \ldots, \lambda_n\}$ and $\sigma(H(\eta_1, \ldots, \eta_{n-2}, \rho_{n-1})) = \{\mu_1, \ldots, \mu_{n-1}\}$ with $\rho_{n-1} = (\eta_{n-1} + \bar{\mu}_0\eta_n)/(1 + \bar{\mu}_0\bar{\eta}_{n-1}\eta_n)$.

Just like JIEPs, the proofs of existence for the above results can be turned into numerical methods, such as the Lanczos/Arnoldi algorithm. More details can be found in Ammar *et al.* (1991) and Ammar and He (1995).

9. Inverse eigenvalue problems with prescribed entries

A very large class of SIEPs can be described as inverse eigenvalue problems with prescribed entries (**PEIEP**). The prescribed entries are used to characterize the underlying structure. The most general setting can be delineated as follows (Ikramov and Chugunov 2000). Given a certain subset $\mathcal{L} = \{(i_t, j_t)\}_{t=1}^{\ell}$ of pairs of subscripts, a certain set of values $\{a_1, \ldots, a_\ell\}$ over a field $\mathbb{F}$, and a set of n values $\{\lambda_1, \ldots, \lambda_n\}$, find a matrix $X \in \mathbb{F}^{n\times n}$ such that

$$\begin{cases} \sigma(X) &= \{\lambda_1, \ldots, \lambda_n\}, \\ X_{i_t,j_t} &= a_t \text{ for } t = 1, \ldots, \ell. \end{cases}$$

Let the cardinality ℓ of the index set $\mathcal{L}$ be denoted by $|\mathcal{L}|$. The PEIEP is to determine (complete) values for the $n^2 - |\mathcal{L}|$ positions that do not belong to $\mathcal{L}$ so as to satisfy the spectral constraint. The Jacobi structure can be considered as a special case of PEIEP where, in addition to the desired symmetry of the band, elements outside the tridiagonal band are required to be zero. Another interesting variation of the PEIEP is the completion problem, where only a one-to-one correspondence between the ℓ positions in $\mathcal{L}$ and the ℓ prescribed values $\{a_1, \ldots, a_\ell\}$, but not in any specific order, is required. Two major points that have been the focus of discussion in the literature are to determine the cardinality $|\mathcal{L}|$ so that the problem makes sense, and to study the effect of the locations in $\mathcal{L}$.

9.1. Prescribed entries along the diagonal

Perhaps a natural place to begin the discussion of PEIEPs is the construction of a Hermitian matrix with prescribed diagonal entries and eigenvalues. Recall that a vector $\mathbf{a} \in \mathbb{R}^n$ is said to majorize $\boldsymbol{\lambda} \in \mathbb{R}^n$ if, assuming the orderings $a_{j_1} \le \cdots \le a_{j_n}$ and $\lambda_{m_1} \le \cdots \le \lambda_{m_n}$ of their elements, the following relationships hold:

$$\begin{cases} \sum_{i=1}^{k} \lambda_{m_i} \le \sum_{i=1}^{k} a_{j_i}, & k = 1, \ldots, n, \\ \sum_{i=1}^{n} \lambda_{m_i} = \sum_{i=1}^{n} a_{j_i}. \end{cases} \tag{9.1}$$

The necessary and sufficient relationship between the diagonal entries and the eigenvalues of a Hermitian matrix is completely characterized by the Schur–Horn theorem.

Theorem 9.1. (Horn 1954*a*) A Hermitian matrix H with eigenvalues $\boldsymbol{\lambda}$ and diagonal entries $\mathbf{a}$ exists if and only if $\mathbf{a}$ *majorizes* $\boldsymbol{\lambda}$.

The sufficient condition is the harder part of the proof and that is precisely the heart of the Schur–Horn inverse eigenvalue problem (**SHIEP**): given two

vectors $\mathbf{a}$ and $\boldsymbol{\lambda}$ where $\mathbf{a}$ majorizes $\boldsymbol{\lambda}$, construct a Hermitian matrix with diagonals $\mathbf{a}$ and eigenvalues $\boldsymbol{\lambda}$. The original proof was done by mathematical induction. A continuous method was discussed in Chu (1995). A finite iterative method was described by Zha and Zhang (1995).

Without the Hermitian structure, the connection between eigenvalues and diagonal entries of a general matrix is given by the Mirsky theorem.

Theorem 9.2. (Mirsky 1958) A matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and main diagonal elements $a_1, \ldots, a_n$ exists if and only if

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \lambda_i. \tag{9.2}$$

Again, the sufficient condition in the Mirsky theorem constitutes an inverse problem where the prescribed entries are along the diagonal. An inverse problem as such would be of little interest. Later, de Oliveira (1973*a*, 1973*b*, 1975) generalized the Mirsky theorem to the case of non-principal diagonals. Given a permutation ϱ, the positions in a matrix corresponding to the index set $\mathcal{D} = \{(i, \varrho(i))\}_{i=1}^{n}$ is called a ϱ-diagonal of that matrix.

Theorem 9.3. (de Oliveira 1973*b*) Let $\{\lambda_1, \ldots, \lambda_n\}$ and $\{a_1, \ldots, a_n\}$ be two sets of arbitrary numbers over a field $\mathbb{F}$. Suppose that at least one of the disjoint cycles in the product representation $\varrho = \varrho_1 \cdots \varrho_s$ has length > 2. Then there exists a matrix $X \in \mathbb{F}^{n\times n}$ such that $\sigma(X) = \{\lambda_1, \ldots, \lambda_n\}$ and $X_{i,\varrho(i)} = a_i$ for $i = 1, \ldots, n$.

The assumption in the de Oliveira theorem, that at least one cycle has length greater than 2, precludes the case that ϱ is the identity and, hence, the equality (9.2) is not needed. If no cycle is of length > 2, then a similar result holds under some additional restrictions (de Oliveira 1973*b*, Theorem 2). Using the so-called L-transform, the proof of the de Oliveira theorem is constructive, and can be converted into a finite numerical algorithm. We shall comment more on this in the next few sections.

Having described two types of PEIEP arising from the Schur–Horn theorem and the de Oliveira theorem, respectively, we might as well bring forth another class of inverse problem with prescribed entries that involves singular values instead of eigenvalues. The following Sing–Thompson theorem characterizes the relationship between singular values and diagonal entries of a general matrix.

Theorem 9.4. (Sing 1976, Thompson 1977) Assume that elements in two given vectors $\mathbf{d}, \mathbf{s} \in \mathbb{R}^n$ satisfy $s_1 \geq s_2 \geq \cdots \geq s_n$ and $|d_1| \geq |d_2| \geq \cdots \geq |d_n|$. Then a real matrix with singular values $\mathbf{s}$ and main diagonal

entries **d** (possibly in different order) exists if and only if

$$\begin{cases} \sum_{i=1}^{k} |d_i| & \leq \sum_{i=1}^{k} s_i, \quad \text{for } k = 1, \ldots, n, \\ \left(\sum_{i=1}^{n-1} |d_i|\right) - |d_n| & \leq \left(\sum_{i=1}^{n-1} s_i\right) - s_n. \end{cases} \tag{9.3}$$

The sufficient condition in the Sing–Thompson theorem gives rise to an inverse singular value problem (**STISVP**) of constructing a square matrix with prescribed diagonals and singular values. The original proof was done by mathematical induction. Chu (1999) rewrote it as a divide-and-conquer algorithm that can easily be implemented in any programming language that supports recursion.

9.2. Prescribed entries at arbitrary locations

Note that the PEIEP in the Mirsky theorem really involves only $n-1$ prescribed entries $a_1, \ldots, a_{n-1}$ because a_n is determined from (9.2). London and Minc (1972) showed that the restriction of the $n-1$ prescribed entries to the main diagonal was unnecessary.

Theorem 9.5. (London and Minc 1972) Given two sets $\{\lambda_1, \ldots, \lambda_n\}$ and $\{a_1, \ldots, a_{n-1}\}$ of arbitrary numbers in an algebraically closed field $\mathbb{F}$, suppose $\mathcal{L} = \{(i_t, j_t)\}_{t=1}^{n-1}$ is a set of arbitrary but distinct positions. Then there exists a matrix $X \in \mathbb{F}^{n\times n}$ such that $\sigma(X) = \{\lambda_1, \ldots, \lambda_n\}$ and $X_{i_t,j_t} = a_t$ for $t = 1, \ldots, n-1$.

An alternative proof was given in de Oliveira (1973*b*). Both proofs used mathematical induction. In principle, we think a fast recursive algorithm similar to those for the SHIEP and the STISVP could be devised. We have not yet seen its numerical implementation. Similar inverse problems constructing matrices with arbitrary $n-1$ prescribed entries and prescribed characteristic polynomials are considered in Dias da Silva (1974) and Ikramov and Chugunov (2000).

An interesting follow-up question to the London–Minc theorem is how many more entries of a matrix can be specified while the associated PEIEP is still solvable. Obviously, as we have learned by now, the locations of these prescribed entries also have some effect on the solvability. To help us better grasp the scope of this complicated issue, we first turn our attention to another subclass of PEIEPs before we return to this question in Section 9.4.

9.3. Additive inverse eigenvalue problem

Thus far, we have considered several cases of PEIEPs with small $|\mathcal{L}|$. With $|\mathcal{L}| = n-1$, the London and Minc theorem asserts that the PEIEP is always solvable with no other constraints. With $|\mathcal{L}| = n$, the PEIEP is solvable under some constraints. Indeed, Ikramov and Chugunov (2000,

Section 3b) argued meticulously through various cases to draw the most general conclusion.

Theorem 9.6. (Ikramov and Chugunov 2000) Suppose that the field $\mathbb{F}$ is algebraically closed and that $|\mathcal{L}| = n$. Assume that the following two conditions are met, if they occur:

$$\begin{cases} \text{that (9.2) is satisfied,} & \text{if } \mathcal{L} = \{(i,i)\}_{i=1}^n, \quad \text{or} \\ \text{that } a_i = \lambda_j \text{ for some } j, & \text{if } \mathcal{L} = \{(i,j_t)\}_{t=1}^n \text{ and } a_t = 0 \text{ for all } j_t \neq i. \end{cases}$$

Then the PEIEP is solvable via rational algorithms in $\mathbb{F}$. (A Maple code that generates a solution in closed form has been implemented by Chugunov.)

In both cases, there is plenty of room, that is, $n^2 - |\mathcal{L}|$ free locations, for constructing such a matrix. In contrast, the classical AIEP (see (2.14)) is another type of PEIEP with much less room for free locations. Recall that an AIEP concerns adding a diagonal matrix D to a given matrix A so that $\sigma(A + D)$ has a prescribed spectrum (note that in a more general context D need not be a diagonal, but can be defined by the *complement* to any index set $\mathcal{L}$). In the AIEP, the prescribed entries consist of all off-diagonal elements, and thus $|\mathcal{L}| = n^2 - n$. In this case, the following brilliant result is due to Friedland (1972). See also Friedland (1977).

Theorem 9.7. (Friedland 1977) The AIEP over any algebraically closed field is always solvable. If n is the order of the problem, then there exist at most $n!$ solutions. For almost all given $\{\lambda_1, \ldots, \lambda_n\}$, there are exactly $n!$ solutions.

Somewhere there is a threshold on the cardinality $|\mathcal{L}|$ of prescribed entries that changes the PEIEP from finitely solvable to finitely unsolvable. It is known that the AIEP in general cannot be solved in finitely many steps. The AIEP in which all off-diagonal entries are 1, for example, is not solvable in radicals for $n \geq 5$. The AIEP for a Jacobi matrix with subdiagonal (and superdiagonal) entries 1 is not solvable in radicals even for $n = 4$ (Ikramov and Chugunov 2000). The AIEP has to be solved by other types of numerical methods (Friedland, Nocedal and Overton 1986).

It is critical to the observer that the solvability assured in both Theorem 9.6 and Theorem 9.7 requires that the underlying field $\mathbb{F}$ is algebraically closed. In Chu (1998), such an AIEP was referred to as AIEP3. The AIEP over the field $\mathbb{R}$ of real numbers was referred to as AIEP1, and AIEP2 if the matrix A is real symmetric. The AIEP is not always solvable over $\mathbb{R}$. It is easy to see, for instance, that a necessary condition for the real solvability of AIEP1 is that $\sum_{i\neq j}(\lambda_i - \lambda_j)^2 \geq 2n \sum_{i\neq j} a_{ij}a_{ji}$. For convenience, define $\pi(M) := \|M - \text{diag}(M)\|_\infty$. The separation of prescribed eigenvalues

relative to the size of (prescribed) off-diagonal entries of A renders some sufficient conditions for the real solvability.

Theorem 9.8. Given a set $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_n\}$ of eigenvalues, define the separation of eigenvalues by

$$d(\boldsymbol{\lambda}) := \min_{i \neq j} |\lambda_i - \lambda_j|. \tag{9.4}$$

Then:

(1) if $d(\lambda) > 2\sqrt{3}(\pi(A \circ A))^{1/2}$, then AIEP2 is solvable (Hadeler 1968);

(2) if $d(\lambda) > 4\pi(A)$, then AIEP1 is solvable (de Oliveira 1970).

The above theorem offers no clue on what will happen when the separation $d(\boldsymbol{\lambda})$ is too small. At the extreme case when two eigenvalues coalesce, we have the following result.

Theorem 9.9. (Shapiro 1983, Sun and Qiang 1986) Both AIEP1 and AIEP2 are unsolvable almost everywhere if there are multiple eigenvalues present in $\boldsymbol{\lambda}$.

Up to this point, we have refrained from venturing into discussion on the class of *parametrized problems* in order to remain focused on the *structured problems*. Nevertheless, as we indicated earlier in Figure 3.1, these problems overlap each other. We have come across the class of PIEPs many times in this paper. It is perhaps fitting to at least describe the PIEP and point to some general results in the context of PIEPs. By a PIEP we mean the IEP of determined parameters $c_1, \ldots, c_\ell$ so that the matrix

$$A(c_1, \ldots, c_\ell) = A_0 + \sum_{t=1}^{\ell} c_t A_t, \tag{9.5}$$

where A_t, $t = 0, \ldots, \ell$, are given matrices, have a prescribed spectrum. It is clear that PEIEPs are a special case of PIEPs by identifying $|\mathcal{L}| = \ell$ and $A_t = \mathbf{e}_{i_t}\mathbf{e}_{j_t}^T$, where $\mathbf{e}_k$ denotes the standard kth coordinate vector in $\mathbb{R}^n$ and (i_t, j_t) is the tth pair of indices in $\mathcal{L}$. Using Brouwer's fixed-point theorem, Biegler-König (1981*b*) derived some sufficient conditions for real solvability of more general PIEPs.

We conclude this section with a sensitivity result for AIEP2 and remarks (Xu 1998, Corollary 4.5.5).

Theorem 9.10. (Xu 1998) Suppose D is a solution to AIEP2 with symmetric matrix A and eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. Let the spectral decomposition of $A + D$ be written as $A + D = Q(D)^T \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\} Q(D)$, with

$Q(D) = [q_{ij}(D)] = [\mathbf{q}_1, \ldots, \mathbf{q}_n]$. Define

$$\Omega(D) := [q_{ji}^2(D)],$$
$$b(D) := [\mathbf{q}_1(D)^T A \mathbf{q}_1(D), \ldots, \mathbf{q}_n(D)^T A \mathbf{q}_n(D)]^T.$$

Assume that the matrix $\Omega(D)$ is nonsingular and that the perturbation

$$\delta = \|\lambda - \tilde{\lambda}\|_\infty + \|A - \tilde{A}\|_2$$

is sufficiently small. Then the AIEP2 associated with the perturbed data $\tilde{A}$ and $\tilde{\lambda}$ is solvable. Furthermore, for the perturbed problem there is a solution $\tilde{D}$ near to D in the sense that

$$\frac{\|D - \tilde{D}\|_\infty}{\|D\|_\infty} \le \kappa_\infty(\Omega(D)) \left(\frac{\|\lambda - \tilde{\lambda}\|_\infty + \|A - \tilde{A}\|_2}{\|\lambda - b\|_\infty} \right) + O(\delta^2),$$

where $\kappa_\infty(M)$ stands for the condition number of the matrix M in the infinity norm.

Observe that PEIEPs, including AIEP2, generally have multiple solutions. The above theorem only ensures that, for a given D, there exists in theory a solution $\tilde{D}$ to the perturbed problem. However, the numerical solution $\tilde{\tilde{D}}$ obtained by a computational method, could be very different from D.

9.4. Cardinality and locations

The prescribed entries in the SHIEP and the STISVP are required to be on the diagonal. So certain inequalities (Theorems 9.1 and 9.4) involving the prescribed eigenvalues and entries must be satisfied. The prescribed entries in an AIEP are required to be on the off-diagonal. Complex solvability was addressed in Theorem 9.7, but real solvability is only partially understood. In all these cases, the prescribed entries are located at special positions.

Theorem 9.6 relaxes the specification to arbitrary locations and, under very mild conditions, asserts the existence of a solution to the PEIEP when $|\mathcal{L}| = n$. It is natural to ask what is the interplay between cardinality and locations so that a PEIEP is solvable. To that end, we describe what is possibly the strongest result on $|\mathcal{L}|$ in the class of PEIEPs at arbitrary locations. The original work was presented in the MSc thesis by Hershkowits (1978). We restate the result from Hershkowits (1983).

Theorem 9.11. (Hershkowits 1983) Suppose that the field $\mathbb{F}$ is algebraically closed and that $|\mathcal{L}| = 2n - 3$. Assume that the following two conditions are met, if they occur:

$$\begin{cases} \text{that (9.2) is satisfied,} & \text{if } \mathcal{L} \supseteq \{(i,i)\}_{i=1}^n, \quad \text{or} \\ \text{that } a_i = \lambda_j \text{ for some } j, & \text{if } \mathcal{L} \supseteq \{(i,j_t)\}_{t=1}^n \text{ and } a_t = 0 \text{ for all } j_t \ne i. \end{cases}$$

Then the PEIEP is solvable in $\mathbb{F}$.

Note that the effect of locations of positions in $\mathcal{L}$ is limited to the two necessary conditions stated in the theorem, and are quite general. The proof of the Hershkowits theorem was established by induction. In principle, it was declared in Ikramov and Chugunov (2000) that the construction could be done by a rational algorithm. The seven basic cases plus the many subcases of analysis in the 15-page proof might make a computer implementation quite a challenge. It would be interesting to see if other numerical algorithms could be developed.

10. Inverse singular value problems

The notion of IEPs can naturally be extended to the inverse singular value problems (**ISVP**). An ISVP concerns the construction of a structured matrix with prescribed singular values. Once again, an ISVP should also satisfy a certain structural constraint. To our knowledge, the class of ISVPs is an entirely new territory that has barely been explored in the literature. Adding to the complication is that the underlying matrix should not be symmetric (otherwise, it is reduced to an IEP) and could be rectangular. We have already seen one type of ISVP, that is, the STISVP in Section 9.1, where a matrix is to be constructed with prescribed diagonal entries and singular values. Another type of ISVP was mentioned in Section 2.5, where a given matrix was to be conditioned by a rank-one perturbation. Clearly, every other type of IEP, except for the symmetric problems, has a counterpart under the context of ISVP.

One of the reasons that we include ISVPs in the context of SIEPs is that an ISVP, even without any string of structure, can be converted into an SIEP. Note that eigenvalues of the *structured* symmetric matrix

$$C = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \tag{10.1}$$

are precisely the pluses and minuses of singular values of B. The IEP for C has the fixed structure of zero diagonal blocks plus whatever structure inherited from B. An ISVP for a structured B is solvable if and only if an IEP for C with structure defined in (10.1) is solvable. To establish conditions on the solvability of a structured ISVP should be an interesting question for further research.

To introduce the notion of ISVPs, we shall limit our discussion to a special class of parametrized ISVPs. Given general matrices $B_0, B_1, \ldots, B_n \in \mathbb{R}^{m\times n}$, $m \geq n$ and nonnegative real numbers $s_1 \geq \cdots \geq s_n$, find values of $c := (c_1, \ldots, c_n)^T \in \mathbb{R}^n$ such that singular values of the matrix

$$B(c) := B_0 + \sum_{i=1}^{n} c_i B_i \tag{10.2}$$

are precisely $\{s_1, \ldots, s_n\}$. In analogy to the PIEP (9.5), the matrices B_i can be used to delineate certain quite general structures. We have already discussed a Newton-type iterative procedure for the ToIEP. We now demonstrate how the ISVP can be handled in a similar but more subtle way. The subtlety comes from the fact that, for ISVPs, we have to deal with the left and the right singular vectors at the same time.

For illustration purposes, assume all prescribed singular values $s_1, \ldots, s_n$ are positive and distinct. Let $\Sigma \in \mathbb{R}^{m\times n}$ denote the 'diagonal' matrix with diagonal elements $\{s_1, \ldots, s_n\}$. Define the affine subspace $\mathcal{B} := \{B(c)|c \in \mathbb{R}^n\}$ and the manifold $\mathcal{M}_s(\Sigma) := \{U\Sigma V^T|U \in \mathcal{O}(m), V \in \mathcal{O}(n)\}$ of all matrices with singular values $\{s_1, \ldots, s_n\}$, where $\mathcal{O}(n)$ is the set of all orthogonal matrices in $\mathbb{R}^{n\times n}$. Solving the ISVP is equivalent to finding an intersection of the two sets $\mathcal{M}_s(\Sigma)$ and $\mathcal{B}$. Note that any tangent vector $T(X)$ to $\mathcal{M}_s(\Sigma)$ at $X \in \mathcal{M}_s(\Sigma)$ must be of the form

$$T(X) = XK - HX,$$

for some skew-symmetric matrices $H \in \mathbb{R}^{m\times m}$ and $K \in \mathbb{R}^{n\times n}$. From any given $X^{(\nu)} \in \mathcal{M}_s(\Sigma)$, factorized as

$${U^{(\nu)}}^T X^{(\nu)} V^{(\nu)} = \Sigma$$

with $U^{(\nu)} \in \mathcal{O}(m)$ and $V^{(\nu)} \in \mathcal{O}(n)$, our goal is twofold. First, we seek for a $\mathcal{B}$-intercept $B(c^{(\nu+1)})$ from a line that is tangent to the manifold $\mathcal{M}_s(\Sigma)$ at $X^{(\nu)}$. Then we seek for a way to lift the matrix $B(c^{(\nu+1)}) \in \mathcal{B}$ to a point $X^{(\nu+1)} \in \mathcal{M}_s(\Sigma)$.

To determine the intercept, we calculate skew-symmetric matrices $H^{(\nu)} \in \mathbb{R}^{m\times m}$ and $K^{(\nu)} \in \mathbb{R}^{n\times n}$, and a vector $c^{(\nu+1)} \in \mathbb{R}^n$ so that the equation

$$X^{(\nu)} + X^{(\nu)}K^{(\nu)} - H^{(\nu)}X^{(\nu)} = B(c^{(\nu+1)}) \tag{10.3}$$

is satisfied. Equivalently, we calculate skew-symmetric matrices $\tilde{H}^{(\nu)} := {U^{(\nu)}}^T H^{(\nu)} U^{(\nu)}$ and $\tilde{K}^{(\nu)} := {V^{(\nu)}}^T K^{(\nu)} V^{(\nu)}$ for the equation

$$\Sigma + \Sigma\tilde{K}^{(\nu)} - \tilde{H}^{(\nu)}\Sigma = \underbrace{{U^{(\nu)}}^T B(c^{(\nu+1)}) V^{(\nu)}}_{W^{(\nu)}}. \tag{10.4}$$

The values for $c^{(\nu+1)}$, $H^{(\nu)}$, and $K^{(\nu)}$ can be determined separately.

Observe that, in total, there are $m(m-1)/2 + n(n-1)/2 + n$ unknowns and mn equations involved in (10.4). A closer examination of (10.4) shows that the lower-right corner of size $(m-n)\times(m-n)$ in $\tilde{H}^{(\nu)}$ can be arbitrary. For simplicity, we set this part to be identically zero. Then it suffices to consider the mn equations

$$W_{ij}^{(\nu)} = \Sigma_{ij} + \Sigma_{ii}\tilde{K}_{ij}^{(\nu)} - \tilde{H}_{ij}^{(\nu)}\Sigma_{jj}, \quad 1 \le i \le m, \quad 1 \le j \le n, \tag{10.5}$$

where $\tilde{K}_{ij}^{(\nu)}$ is understood to be zero if $i \geq n+1$, for the remaining quantities.

For $1 \leq i = j \leq n$, we obtain

$$\Omega^{(\nu)} c^{(\nu+1)} = \mathbf{s} - \mathbf{b}^{(\nu)}, \tag{10.6}$$

where

$$\begin{aligned} \Omega_{st}^{(\nu)} &:= {\mathbf{u}_s^{(\nu)}}^T B_t \mathbf{v}_s^{(\nu)}, \quad 1 \leq s, t \leq n, \\ \mathbf{s} &:= [s_1, \ldots, s_n]^T, \quad \text{and} \\ b_s^{(\nu)} &:= {\mathbf{u}_s^{(\nu)}}^T B_0 \mathbf{v}_s^{(\nu)}, \quad 1 \leq s \leq n, \end{aligned}$$

if $\mathbf{u}_s^{(\nu)}$ and $\mathbf{v}_s^{(\nu)}$ denote column vectors of $U^{(\nu)}$ and $V^{(\nu)}$, respectively. Under mild assumptions, the matrix $\Omega^{(\nu)}$ is nonsingular. The vector $c^{(\nu+1)}$ and, hence, the matrix $W^{(\nu)}$ are thus obtained.

The skew-symmetric matrices $H^{(\nu)}$ and $K^{(\nu)}$ can be obtained by comparing the 'off-diagonal' entries in (10.4) without much trouble. For $n+1 \leq i \leq m$ and $1 \leq j \leq n$, it is clear that

$$\tilde{H}_{ij}^{(\nu)} = -\tilde{H}_{ji}^{(\nu)} = -\frac{W_{ij}^{(\nu)}}{s_j}. \tag{10.7}$$

For $1 \leq i < j \leq n$,

$$\tilde{H}_{ij}^{(\nu)} = -\tilde{H}_{ji}^{(\nu)} = \frac{s_i W_{ji}^{(\nu)} + s_j W_{ij}^{(\nu)}}{s_i^2 - s_j^2}, \tag{10.8}$$

$$\tilde{K}_{ij}^{(\nu)} = -\tilde{K}_{ji}^{(\nu)} = \frac{s_i W_{ij}^{(\nu)} + s_j W_{ji}^{(\nu)}}{s_i^2 - s_j^2}. \tag{10.9}$$

The intercept is now completely determined.

It only remains to lift the intercept $B(c^{(\nu+1)}$ back to $\mathcal{M}_s(\Sigma)$. Towards that end, one possible way is to define the lift as

$$X^{(\nu+1)} := R^T X^{(\nu)} S,$$

where R and S are the Cayley transforms

$$\begin{aligned} R &:= \left(I + \frac{H^{(\nu)}}{2}\right)\left(I - \frac{H^{(\nu)}}{2}\right)^{-1}, \\ S &:= \left(I + \frac{K^{(\nu)}}{2}\right)\left(I - \frac{K^{(\nu)}}{2}\right)^{-1}. \end{aligned}$$

This completes one cycle of the Newton step and the iteration repeats until convergence. Regarding the efficiency of this algorithm, Chu (1992) proved the following result on the rate of convergence.

Theorem 10.1. Suppose that the ISVP (10.2) has an exact solution at c^* and that $B(c^*) = \hat{U}\Sigma\hat{V}^T$ is the corresponding singular value decomposition.

Define the error matrix $E := (E_1, E_2) := (U - \hat{U}, V - \hat{V})$, and suppose that the matrix $\Omega^{(\nu)}$ is nonsingular. Then $\|E^{(\nu+1)}\| = O(\|E^{(\nu)}\|^2)$

We conclude this section with one important remark concerning the case of multiple singular values. Similar remarks might be applicable to IEPs with prescribed multiple eigenvalues as well. It is known that multiple eigenvalues are difficult to compute even in direct problems. Recall earlier in Theorem 9.9 that AIEP1 and AIEP2 are unsolvable almost everywhere in the presence of multiple eigenvalues. Thus, a general rule of thumb is that the ISVP (IEP) may not have a solution if there are repeated singular values (eigenvalues) in the prescribed set. At least, the proximity of multiple spectral data would impose considerable difficulty to the inverse problem. An argument was given in Chu (1992) showing that only a portion of the total set of singular values of $B(c)$ should be specified to give leeway to accommodate the multiplicity.

11. Inverse singular/eigenvalue problems

The structure involved in SIEPs can be quite general. Thus far, we have seen structures including Jacobi, Toeplitz, nonnegative, stochastic, unitary Hessenberg, prescribed entries, and the special form (10.1). Most of these structural constraints are explicitly or, at least, semi-explicitly, given in terms of the appearance of the underlying matrix. It is possible that the structure is described implicitly as the solution set of some nonlinear functions. In this section, we discuss one particular class of SIEPs where the 'structure' is implicitly characterized by the singular values.

Recall that the Schur–Horn theorem identifies the connection between diagonal entries and eigenvalues of a Hermitian matrix. The Mirsky theorem gives the connection between diagonal entries and eigenvalues of a general matrix. The Sing–Thompson theorem characterizes the connection between diagonal entries and singular values of a general matrix. It is natural to ask about the connection between singular values and eigenvalues of a matrix. For Hermitian matrices, the singular values are simply the absolute values of eigenvalues. But for general square matrices, the connection is much more involved, as is given by the Weyl–Horn theorem.

Theorem 11.1. (Weyl 1949, Horn 1954*b*) Given vectors $\boldsymbol{\lambda} \in \mathbb{C}^n$ and $\mathbf{s} \in \mathbb{R}^n$, suppose the entries are arranged in the ordering that $|\lambda_1| \geq \cdots \geq |\lambda_n|$ and $s_1 \geq \cdots \geq s_n$. Then a matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and singular values $s_1, \ldots, s_n$ exists if and only if

$$\begin{cases} \prod_{j=1}^{k} |\lambda_j| \leq \prod_{j=1}^{k} s_j, & k = 1, \ldots, n-1, \\ \prod_{j=1}^{n} |\lambda_j| = \prod_{j=1}^{n} s_j. \end{cases} \tag{11.1}$$

If $|\lambda_n| > 0$, then the Weyl–Horn condition is equivalent to the statement that the vector $\log(\mathbf{s})$ majorizes the vector $\log|\boldsymbol{\lambda}|$. The IEP we are concerned with is to construct a matrix with prescribed singular values and eigenvalues (**ISEP**), if the Weyl–Horn condition is met. The original proof was by induction, but Chu (2000) modified the proof to avoid triangularization and derived a divide-and-conquer recursive algorithm, which we outline below.

First, observe that the 2×2 triangular matrix

$$A = \begin{bmatrix} \lambda_1 & \mu \\ 0 & \lambda_2 \end{bmatrix}$$

has singular value $\{s_1, s_2\}$ if and only if

$$\mu = \sqrt{s_1^2 + s_2^2 - |\lambda_1|^2 - |\lambda_2|^2}.$$

The fact that μ is well defined follows from the Weyl–Horn condition that $|\lambda_1| \leq s_1$ and $|\lambda_1|\,|\lambda_2| = s_1 s_2$. It is interesting to note that μ^2 is precisely the so-called departure of A from normality. For the sake of better computational stability, we suggest replacing μ by the definition

$$\mu = \begin{cases} 0, & \text{if } |(s_1 - s_2)^2 - (|\lambda_1| - |\lambda_2|)^2| \leq \epsilon, \\ \sqrt{|(s_1 - s_2)^2 - (|\lambda_1| - |\lambda_2|)^2|}, & \text{otherwise.} \end{cases}$$

The 2×2 matrix serves as the building block in the recursion.

The basic ideas in Weyl–Horn's proof contain three major components:

- the original problem can be reduced to two problems of smaller sizes,
- problems of smaller sizes are guaranteed to be solvable by the induction hypothesis, and
- the subproblems can be affixed together by working on a suitable 2×2 corner that has an explicit solution.

If we repeatedly apply these principles, then the original inverse problem is *divided* into subproblems of size 2×2 or 1×1 that can eventually be *conquered* to build up the original size.

The original idea on how the problems could be divided is quite intriguing. Since this is an entirely new approach different from either the iterative methods or the continuous methods we have discussed thus far for other types of IEPs, we outline the proof as follows. For simplicity, we assume that $s_i > 0$ for all $i = 1, \ldots, n$. It follows that $\lambda_i \neq 0$ for all i. The case of zero singular values can be handled in a similar way. Starting with $\gamma_1 := s_1$, define the sequence

$$\gamma_i := \gamma_{i-1} \frac{s_i}{|\lambda_i|}, \quad i = 2, \ldots, n-1. \tag{11.2}$$

Assume that the maximum $\gamma := \min_{1\le i\le n-1}\gamma_i$ is attained at the index j. Define

$$\theta := \frac{|\lambda_1\lambda_n|}{\gamma}. \tag{11.3}$$

Then the following three sets of inequalities are valid:

$$\begin{cases} |\lambda_1| \ge |\lambda_n|, \\ \gamma \ge \theta; \end{cases} \tag{11.4}$$

$$\begin{cases} \gamma \ge |\lambda_2| \ge \cdots \ge |\lambda_j|, \\ s_1 \ge s_2 \ge \cdots \ge s_j; \end{cases} \tag{11.5}$$

$$\begin{cases} |\lambda_{j+1}| \ge \cdots \ge |\lambda_{n-1}| \ge \theta, \\ s_{j+1} \ge \cdots \ge s_{n-1} \ge s_n. \end{cases} \tag{11.6}$$

More importantly, the numbers in each of the above sets satisfy the Weyl–Horn condition, respectively, with the first row playing the role of eigenvalues and the second row playing the singular values. Since these are problems of *smaller* sizes, by induction hypothesis, the ISEPs associated with (11.5) and (11.6) are solvable. In particular, there exist unitary matrices U_1, $V_1 \in \mathbb{C}^{j\times j}$ and *triangular* matrices A_1 such that

$$U_1 \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & & 0 \\ \vdots & \ddots & & \\ 0 & 0 & \cdots & s_j \end{bmatrix} V_1^* = A_1 = \begin{bmatrix} \gamma & \times & \times & \cdots & \times \\ 0 & \lambda_2 & & & \times \\ & & & & \\ \vdots & & & \ddots & \\ 0 & 0 & & & \lambda_j \end{bmatrix},$$

and unitary matrices U_2, $V_2 \in \mathbb{C}^{(n-j)\times(n-j)}$, and *triangular* matrix A_2 such that

$$U_2 \begin{bmatrix} s_{j+1} & 0 & \cdots & 0 \\ 0 & s_{j+2} & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & s_n \end{bmatrix} V_2^* = A_2 = \begin{bmatrix} \lambda_{j+1} & \times & \cdots & \times & \times \\ 0 & \lambda_{j+2} & & & \times \\ \vdots & & \ddots & & \vdots \\ & & & \lambda_{n-1} & \times \\ 0 & 0 & \cdots & 0 & \theta \end{bmatrix}.$$

Note the positions of γ and θ in the matrices. If we augment A_1 and A_2 to

$$\begin{bmatrix} A_1 & \bigcirc \\ \bigcirc & A_2 \end{bmatrix}, \tag{11.7}$$

then γ and θ reside, respectively, at the $(1,1)$ and the (n,n) positions. In

his original proof, Horn claimed that the block matrix could be *permuted* to the triangular matrix

$$\begin{bmatrix}
\lambda_2 & \times & \cdots & \times & \times & & & & & \\
0 & & & & \times & & & & & \\
\vdots & & \ddots & & \vdots & & & \bigcirc & & \\
 & & & \lambda_j & \times & & & & & \\
0 & & \cdots & 0 & \gamma & 0 & & & & \\
0 & 0 & \cdots & 0 & 0 & \theta & \times & \times & \cdots & \times \\
 & & & & & & \lambda_{j+1} & & & \times \\
 & & \bigcirc & & & & & & & \\
 & & & & & \vdots & & & \ddots & \\
 & & \cdots & & & 0 & 0 & & & \lambda_{n-1}
\end{bmatrix}.$$

but this is not quite correct. If it were true, it is obvious that the resulting matrix would have singular values $\{s_1, \ldots, s_n\}$ and miss only the eigenvalues $\{\lambda_1, \lambda_n\}$. The next step is to glue the 2×2 corner adjacent to the two blocks together by an equivalence transformation

$$U_0 \begin{bmatrix} \gamma & 0 \\ 0 & \theta \end{bmatrix} V_0^* = A_0 = \begin{bmatrix} \lambda_1 & \mu \\ 0 & \lambda_n \end{bmatrix}$$

that does not affect the eigenvalues $\{\lambda_2, \ldots, \lambda_{n-1}\}$.

In Horn's proof, the ordering of diagonal entries is important and the resulting matrix is upper-triangular. While the final result in the Schur–Horn theorem remains true, it is unfortunate that it takes more than permutations to rearrange the diagonals of a triangular matrix while maintaining the singular values. Such a rearrangement is needed at every conquering step, but it requires new Schur decompositions and is expensive to compute in general.

It was proved in Chu (2000) that the triangular structure was entirely unnecessary, as was the rearrangement of the diagonal entries. It can be shown that modifying the first and the last rows and columns of the block diagonal matrix in (11.7) is sufficient to solve the ISEP, and that the resulting matrix is permutation similar to a triangular matrix. This advance in understanding makes it possible to implement the induction proof as a numerical algorithm.

More precisely, denote the 2×2 orthogonal matrices by $U_0 = [u_{st}^{(0)}]$ and $V_0 = [v_{st}^{(0)}]$. Then the matrix

$$A = \begin{bmatrix} u_{11}^{(0)} & 0 & u_{12}^{(0)} \\ 0 & I_{n-1} & 0 \\ u_{21}^{(0)} & 0 & u_{22}^{(0)} \end{bmatrix} \begin{bmatrix} A_1 & \bigcirc \\ \bigcirc & A_2 \end{bmatrix} \begin{bmatrix} v_{11}^{(0)} & 0 & v_{12}^{(0)} \\ 0 & I_{n-1} & 0 \\ v_{21}^{(0)} & 0 & v_{22}^{(0)} \end{bmatrix}^*$$

has eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ and singular values $\{s_1, \ldots, s_n\}$. The resulting matrix A has the structure

$$A = \begin{bmatrix}
\lambda_1 & \otimes & \cdots & \otimes & \otimes & * & * & & & \mu \\
\otimes & \lambda_2 & & & \times & 0 & 0 & & & * \\
\vdots & & \ddots & & \vdots & & & \bigcirc & & \\
 & & & \lambda_{j-1} & \times & & & & & \\
\otimes & \times & \cdots & \times & \lambda_j & & & & & * \\
* & 0 & \cdots & 0 & 0 & \lambda_{j+1} & \times & \times & \cdots & \otimes \\
* & & & & 0 & \times & \lambda_{j+2} & & & \otimes \\
 & & \bigcirc & & & & & & & \\
 & & & & & \vdots & & & \ddots & \\
0 & * & \cdots & * & * & \otimes & \otimes & & & \lambda_n
\end{bmatrix},$$

where $\times$ stands for unchanged, original entries from A_1 or A_2, $\otimes$ stands for entries of A_1 or A_2 that are modified by scalar multiplications, and $*$ denotes possible new entries that were originally zero. This pattern repeats itself during the recursion. Note that diagonal entries of A_1 and A_2 are in the fixed orders $\gamma, \lambda_2, \ldots, \lambda_j$ and $\lambda_{j+1}, \ldots, \lambda_{n-1}, \theta$, respectively. Each A_i is similar via permutations, which need not be known, to a lower-triangular matrix whose diagonal entries constitute the same set as the diagonal entries of A_i. Thus the eigenvalues of each A_i are precisely its diagonal entries. The first row and the last row have the same zero pattern except that the lower-left corner is always zero. The first column and the last column have the same zero pattern except that the lower-left corner is always zero. Using graph theory, it can be shown that the affixed matrix A has exactly the same properties.

With this realization, the entire induction process can easily be implemented in any programming language that allows a routine to call itself recursively. The main feature in the routine should be a single divide-and-conquer mechanism as we just described. As the routine is calling itself recursively, the problem is 'divided down' and 'conquered up' accordingly. A sample MATLAB program can be found in Chu (2000).

We illustrate how the divide-and-conquer algorithm works by a 6×6 symbolic example. In the following, the integers j_ℓ, selected randomly only for demonstration, indicate where the problem should be divided.

The dividing process, along with the corresponding eigenvalues and singular values for each subproblems, is depicted in the boxed frames in Figure 11.1. A blank framed box indicates that the division has reached the bottom. In this example, the original 6×6 problem is divided into two 1×1 problems and two 2×2 problems. Each of these small problems can trivially be solved. The pair of numbers beside j_ℓ and in between rows of framed

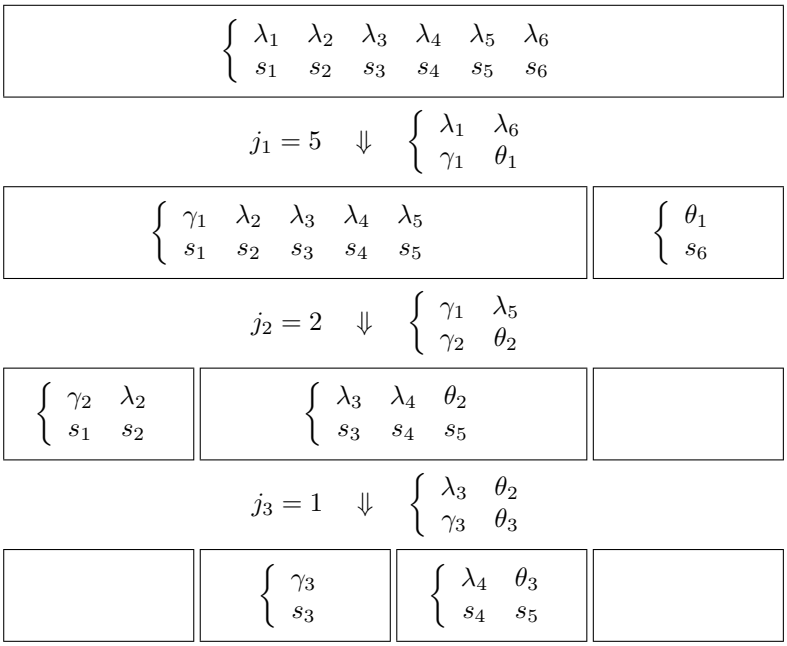

Figure 11.1. An illustration of the dividing process

boxes are the eigenvalues and singular values for the 2×2 matrix used to fasten the smaller matrices together in the conquering process.

The conquering process using the small matrices to build larger matrices is depicted in Figure 11.2. The matrices beside j_ℓ, and in between rows of framed boxes, are the augmented matrices (11.7) with the wrong eigenvalues. After fixing by some appropriated 2×2 matrices, we see in Figure 11.2 that some rows and columns must be modified. The symbols $\times, \otimes$ and $*$, indicating how the values have been changed during the conquering process, have the same meaning as defined before. The final 6×6 matrix with the desirable eigenvalues and singular values has the structure indicated at the top of Figure 11.2.

It is perhaps true that eigenvalues and singular values are two of the most important characteristics of a matrix. Being able to construct a solution for the ISEP might help to create test matrices for numerical linear algebra algorithms.

We should point out promptly that the constructed matrix obtained by the algorithm above is usually complex-valued, if there are complex eigenvalues. It might be desirable to construct a real-valued solution, if all eigenvalues are complete in conjugation. Towards that end, very recently Li and Mathias (2001) extended the Weyl–Horn condition to the case when only $m(\leq n)$ eigenvalues are given. They also proposed a stable algorithm using diagonal

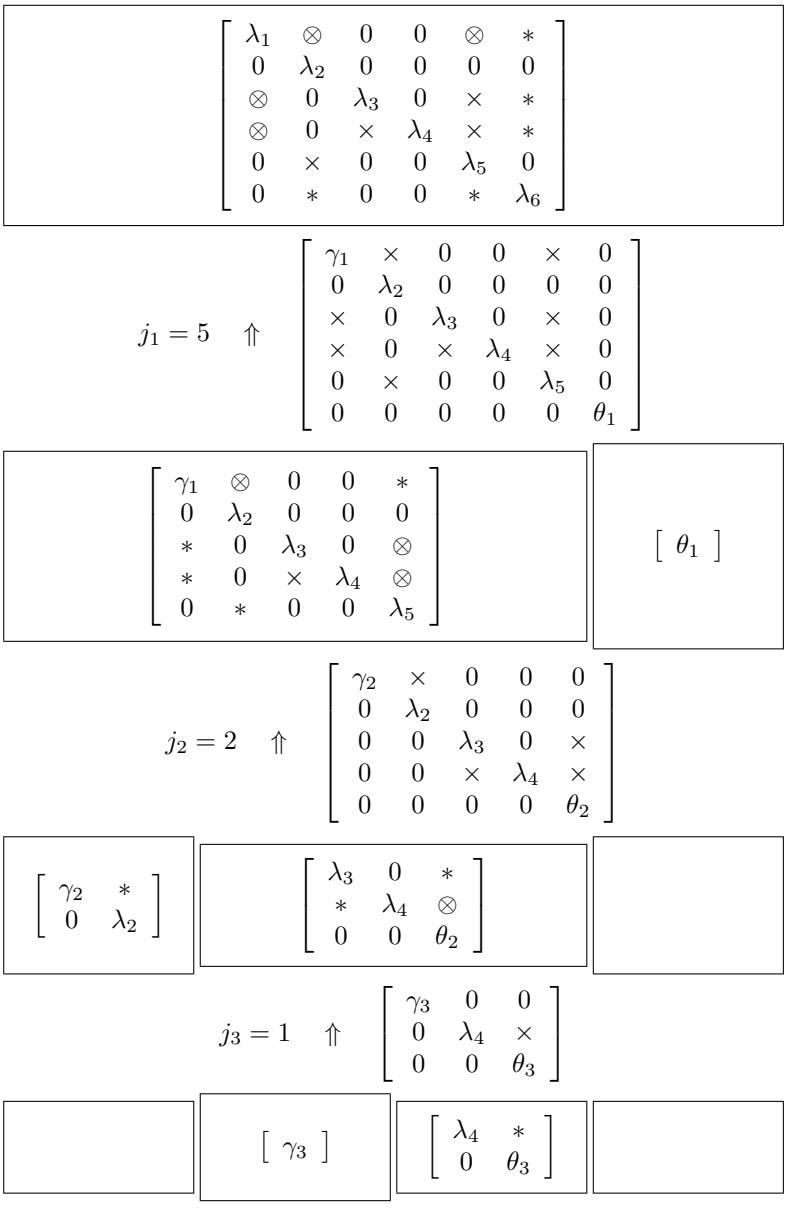

Figure 11.2. An illustration of the conquering process

unitary matrices, permutations, and rotation matrices to construct a real matrix if the specified eigenvalues are closed in complex conjugation. Finally, we remark that the divide-and-conquer feature enables fast computation. Numerical experiments seem to suggest that the overall cost in solving an n-dimensional ISEP is roughly $O(n^2)$.

12. Conclusion

We believe that inverse eigenvalue problems should always be structured problems. In this article, we try to explain, motivate, and review only a small segment of the full scope of structured inverse eigenvalue problems. The structures we selected for study in this presentation are by no means emblematic, but rather reflect our personal preferences. The notions introduced in this paper are by no means conclusive, but rather divulge our limited understanding of this subject. We have collected an extensive bibliography of more than 400 papers on this topic (available at `http://www4.ncsu.edu/~mtchu`). Even with that, the list is far from being comprehensive as we have already overlooked much of the engineering literature. Furthermore, be aware that our consideration has been limited to the setting when the entire spectrum is known and that the structural constraint must be satisfied. We have not discussed the structured problems where only partial eigenvalues and eigenvectors are given. Neither have we examined the case where a least squares solution with approximate spectrum or approximate structure is sufficient for practical purposes.

We hope to have accomplished three goals in this presentation. First, we wanted to demonstrate the breadth of areas where inverse eigenvalue problems can arise. The discipline ranges from practical engineering applications to abstract algebraic theorization. Secondly, we wanted to corroborate the depth of intricacy of inverse eigenvalue problems. While the set-up of an inverse eigenvalue problem seems relatively easy, the solution is not straightforward. The instruments employed to solve such a problem are quite sophisticated, including techniques from orthogonal polynomials, degree theory, optimization, to differential geometry and so on. Finally and most importantly, we wanted to arouse interest and encourage further research into this topic. We have indicated throughout the text that there is much room for further study of the numerical development and theoretical understanding of these fascinating inverse problems.

REFERENCES

M. Adler, L. Haine and P. van Moerbeke (1993), 'Limit matrices for the Toda flow and periodic flags for loop groups', *Math. Ann.* **296**, 1–33.

G. Ammar, W. Gragg and L. Reichel (1991), Constructing a unitary Hessenberg matrix from spectral data, in *Numerical Linear Algebra, Digital Signal Processing and Parallel Algorithms* (Leuven, 1988), Springer, Berlin, pp. 385–395.

G. S. Ammar and C. Y. He (1995), 'On an inverse eigenvalue problem for unitary Hessenberg matrices', *Lin. Alg. Appl.* **218**, 263–271.

S. A. Andrea and T. G. Berry (1992), 'Continued fractions and periodic Jacobi matrices', *Lin. Alg. Appl.* **161**, 117–134.

A. L. Andrew (1994), Some recent developments in inverse eigenvalue problems, in *Computational Techniques and Applications: CTAC93* (D. Stewart, H. Gardner and D. Singleton, eds), World Scientific, Singapore, pp. 94–102.

V. Barcilon (1974*a*), 'On the solution of inverse eigenvalue problems with high orders', *Geophys. J. Royal Astronomical Soc.* **39**, 153–154.

V. Barcilon (1974*b*), 'On the uniqueness of inverse eigenvalue problems', *Geophys. J. Royal Astronomical Soc.* **38**, 287–298.

V. Barcilon (1979), 'On the multiplicity of solutions of the inverse problems for a vibrating beam', *SIAM J. Appl. Math.* **37**, 119–127.

V. Barcilon (1990), 'A two-dimensional inverse eigenvalue problem', *Inverse Problems* **6**, 11–20.

W. W. Barrett and C. R. Johnson (1984), 'Possible spectra of totally positive matrices', *Lin. Alg. Appl.* **62**, 231–233.

A. Berman and R. J. Plemmons (1979), *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York. Also Vol. 9 of *Classics in Applied Mathematics*, SIAM, Philadelphia (1994).

F. W. Biegler-König (1981*a*), 'Construction of band matrices from spectral data', *Lin. Alg. Appl.* **40**, 79–87.

F. W. Biegler-König (1981*b*), 'Sufficient conditions for the solubility of inverse eigenvalue problems', *Lin. Alg. Appl.* **40**, 89–100.

D. Boley and G. H. Golub (1984), 'A modified method for reconstructing periodic Jacobi matrices', *Math. Comput.* **24**, 143–150.

D. Boley and G. H. Golub (1987), 'A survey of matrix inverse eigenvalue problems', *Inverse Problems* **3**, 595–622.

D. L. Boley and G. H. Golub (1978), The matrix inverse eigenvalue problem for periodic Jacobi matrices, in *Proc. Fourth Symposium on Basic Problems of Numerical Mathematics* (Plzeň, 1978), Charles University, Prague, pp. 63–76.

C. de Boor and G. H. Golub (1978), 'The numerically stable reconstruction of a Jacobi matrix from spectral data', *Lin. Alg. Appl.* **21**, 245–260.

M. Boyle and D. Handelman (1991), 'The spectra of nonnegative matrices via symbolic dynamics', *Ann. Math.* **133**, 249–316.

A. Bunse-Gerstner, R. Byers, V. Mehrmann and N. K. Nichols (1991), 'Numerical computation of an analytic singular value decomposition of a matrix valued function', *Numer. Math.* **60**, 1–40.

C. I. Byrnes (1989), Pole placement by output feedback, in *Three Decades of Mathematical Systems Theory*, Vol. 135 of *Lecture Notes in Control and Information Sciences*, Springer, pp. 31–78.

J. A. Cadzow and D. M. Wilkes (1990), Signal enhancement and the SVD, in *Proc. 2nd International Workshop on SVD and Signal Processing*, University of Rhode Island, Kingston, RI, pp. 144–151.

D. Calvetti, G. H. Golub, W. B. Gragg and L. Reichel (2000), 'Computation of Gauss–Kronrod quadrature rules', *Math. Comput.* **69**, 1035–1052.

M. P. Calvo, A. Iserles and A. Zanna (1997), 'Numerical solution of isospectral flows', *Math. Comput.* **66**, 1461–1486.

A. Cantoni and F. Bulter (1976), 'Eigenvalues and eigenvectors of symmetric centrosymmetric matrices', *Lin. Alg. Appl.* **13**, 275–288.

K. Chadan, D. Colton, L. Päivärinta and W. Rundell (1997), *An Introduction to Inverse Scattering and Inverse Spectral Problems*, SIAM, Philadelphia, PA.

M. T. Chu (1992), 'Numerical methods for inverse singular value problems', *SIAM J. Numer. Anal.* **29**, 885–903.

M. T. Chu (1993), On a differential equation $\frac{\mathrm{d}X}{\mathrm{d}t} = [x, k(x)]$ where k is a Toeplitz annihilator. Available at `http://www4.ncsu.edu/~mtchu`.

M. T. Chu (1994), On the refinement of a Newt

M. T. Chu (1994), On the refinement of a Newton method for the inverse Toeplitz eigenvalue problem. Available at `http://www4.ncsu.edu/~mtchu`.

M. T. Chu (1995), 'Constructing a Hermitian matrix from its diagonal entries and eigenvalues', *SIAM J. Matrix Anal. Appl.* **16**, 207–217.

M. T. Chu (1998), 'Inverse eigenvalue problems', *SIAM Rev.* **40**, 1–39 (electronic).

M. T. Chu (1999), 'On constructing matrices with prescribed singular values and diagonal elements', *Lin. Alg. Appl.* **288**, 11–22.

M. T. Chu (2000), 'A fast recursive algorithm for constructing matrices with prescribed eigenvalues and singular values', *SIAM J. Numer. Anal.* **37**, 1004–1020 (electronic).

M. T. Chu and K. R. Driessel (1991), 'Constructing symmetric nonnegative matrices with prescribed eigenvalues by differential equations', *SIAM J. Math. Anal.* **22**, 1372–1387.

M. T. Chu and Q. Guo (1998), 'A numerical method for the inverse stochastic spectrum problem', *SIAM J. Matrix Anal. Appl.* **19**, 1027–1039.

R. M. Corless, P. M. Gianni, B. M. Trager and S. M. Watt (1995), The singular value decomposition for polynomial system, in *Proc. Int. Symp. Symbolic and Algebraic Computation*, Montreal, Canada, pp. 195–207.

H. Dai (1995), 'About an inverse eigenvalue problem arising in vibration analysis', *RAIRO Modél. Math. Anal. Numér.* **29**, 421–434.

B. N. Datta, S. Elhay and Y. M. Ram (1997), 'Orthogonality and partial pole assignment for the symmetric definite quadratic pencil', *Lin. Alg. Appl.* **257**, 29–48.

A. S. Deakin and T. M. Luke (1992), 'On the inverse eigenvalue problem for matrices', *J. Phys. A* **25**, 635–648.

P. Delsarte and Y. Genin (1984), Spectral properties of finite Toeplitz matrices, in *Mathematical Theory of Networks and Systems*, Vol. 58 of *Lecture Notes in Control and Information Sciences*, Springer, pp. 194–213.

J. A. Dias da Silva (1974), 'Matrices with prescribed entries and characteristic polynomial', *Proc. Amer. Math. Soc.* **45**, 31–37.

L. Dieci, R. D. Russel and E. S. V. Vleck (1994), 'Unitary integrators and applications to continuous orthonormalization techniques', *SIAM J. Numer. Anal.* **31**, 261–281.

A. C. Downing and A. S. Householder (1956), 'Some inverse characteristic value problems', *J. Assoc. Comput. Mach.* **3**, 203–207.

A. L. Duarte (1989), 'Construction of acyclic matrices from spectral data', *Lin. Alg. Appl.* **113**, 173–182.

R. Erra and B. Philippe (1997), 'On some structured inverse eigenvalue problems', *Numer. Alg.* **15**, 15–35.

W. Ferguson (1980), 'The construction of Jacobi and periodic Jacobi matrices with prescribed spectra', *Math. Comput.* **35**, 79–84.
M. Fiedler (1974), 'Eigenvalues of nonnegative symmetric matrices', *Lin. Alg. Appl.* **9**, 119–142.
E. Foltete, G. M. L. Gladwell and G. Lallement (2001), 'On the reconstruction of a damped vibrating system from two complex spectra, Part II: Experiment', *J. Sound Vib.* **240**, 219–240.
G. E. Forsythe and E. G. Straus (1955), 'On best conditioned matrices', *Proc. Amer. Math. Soc.* **6**, 340–345.
S. Friedland (1972), 'Matrices with prescribed off-diagonal elements', *Israel J. Math.* **11**, 184–189.
S. Friedland (1977), 'Inverse eigenvalue problems', *Lin. Alg. Appl.* **17**, 15–51.
S. Friedland (1978), 'On an inverse problem for nonnegative and eventually nonnegative matrices', *Israel J. Math.* **29**, 43–60.
S. Friedland and A. A. Melkman (1979), 'On the eigenvalues of nonnegative Jacobi matrices', *Lin. Alg. Appl.* **25**, 239–254.
S. Friedland, J. Nocedal and M. L. Overton (1986), Four quadratically convergent methods for solving inverse eigenvalue problems, in *Numerical Analysis* (Dundee, 1985), Longman Sci. Tech., Harlow, pp. 47–65.
S. Friedland, J. Nocedal and M. L. Overton (1987), 'The formulation and analysis of numerical methods for inverse eigenvalue problems', *SIAM J. Numer. Anal.* **24**, 634–667.
F. R. Gantmaher and M. G. Kreĭn (1960), *Oszillationsmatrizen, Oszillationskerne und Kleine Schwingungen Mechanischer Systeme* (*Oscillation Matrices and Kernels and Small Oscillations of Mechanical Systems*), Berlin, Akademie-Verlag, 1960. (In German.)
I. M. Gel′fand and B. M. Levitan (1955), 'On the determination of a differential equation from its spectral function', *Amer. Math. Soc. Transl.* (2) **1**, 253–304.
G. M. L. Gladwell (1984), 'The inverse problem for the vibrating beam', *Proc. R. Soc.* **A 393**, 277–295.
G. M. L. Gladwell (1986*a*), 'The inverse mode problem for lumped-mass systems', *Quart. J. Mech. Appl. Math.* **39**, 297–307.
G. M. L. Gladwell (1986*b*), *Inverse Problems in Vibration*, Martinus Nijhoff, Dordrecht, Netherlands.
G. M. L. Gladwell (1986*c*), 'Inverse problems in vibration', *Appl. Mech. Review* **39**, 1013–1018.
G. M. L. Gladwell (1996), 'Inverse problems in vibration, II', *Appl. Mech. Review* **49**, 2–27.
G. M. L. Gladwell (1997), 'Inverse vibration problems for finite-element models', *Inverse Problems* **13**, 311–322.
G. M. L. Gladwell (1999), 'Inverse finite element vibration problems', *J. Sound Vib.* **211**, 309–324.
G. M. L. Gladwell (2001), 'On the reconstruction of a damped vibrating system from two complex spectra, Part I: Theory', *J. Sound Vib.* **240**, 203–217.
G. M. L. Gladwell and J. A. Gbadeyan (1985), 'On the inverse problem of the vibrating string or rod', *Quart. J. Mech. Appl. Math.* **38**, 169–174.

G. M. L. Gladwell and A. Morassi (1999), 'Estimating damage in a rod from changes in node position', *Inverse Problems in Engineering* **7**, 215–233.
G. M. L. Gladwell and B. R. Zhu (1992), 'Inverse problems for multidimensional vibrating systems', *Proc. Roy. Soc. London Ser. A* **439**, 511–530.
W. Glunt, T. L. Hayden, S. Hong and J. Wells (1990), 'An alternating projection algorithm for computing the nearest Euclidean distance matrix', *SIAM J. Matrix Anal. Appl.* **11**, 589–600.
J. C. Gower (1982), 'Euclidean distance geometry', *Math. Sci.* **7**, 1–14.
W. B. Gragg and W. J. Harrod (1984), 'The numerically stable reconstruction of Jacobi matrices from spectral data', *Numer. Math.* **44**, 317–335.
L. J. Gray and D. G. Wilson (1976), 'Construction of a Jacobi matrix from spectral data', *Lin. Alg. Appl.* **14**, 131–134.
A. Greenbaum and G. H. Rodrigue (1989), 'Optimal preconditioners of a given sparsity pattern', *BIT* **29**, 610–634.
W. Guo (1996), 'An inverse eigenvalue problem for nonnegative matrices', *Lin. Alg. Appl.* **249**, 67–78.
K. P. Hadeler (1968), 'Ein inverse Eigenwertproblem', *Lin. Alg. Appl.* **1**, 83–101.
O. H. Hald (1972), On discrete and numerical inverse Sturm–Liouville problems, PhD thesis, New York University.
O. H. Hald (1976), 'Inverse eigenvalue problems for Jacobi matrices', *Lin. Alg. Appl.* **14**, 63–85.
P. R. Halmos (1974), *Finite-Dimensional Vector Spaces*, 2nd edn, Springer, New York. Reprinting of the 1958 second edition, *Undergraduate Texts in Mathematics*.
D. Hershkowits (1978), Existence of matrices satisfying prescribed conditions, Master's thesis, Technion, Haifa.
D. Hershkowits (1983), 'Existence of matrices with prescribed eigenvalues and entries', *Lin. Multilin. Alg.* **14**, 315–342.
H. Hochstadt (1967), 'On some inverse problems in matrix theory', *Arch. Math.* **18**, 201–207.
H. Hochstadt (1974), 'On construction of a Jacobi matrix from spectral data', *Lin. Alg. Appl.* **8**, 435–446.
H. Hochstadt (1979), 'On the construction of a Jacobi matrix from mixed given data', *Lin. Alg. Appl.* **28**, 113–115.
A. Horn (1954*a*), 'Doubly stochastic matrices and the diagonal of a rotation matrix', *Amer. J. Math.* **76**, 620–630.
A. Horn (1954*b*), 'On the eigenvalues of a matrix with prescribed singular values', *Proc. Amer. Math. Soc.* **5**, 4–7.
P. Horst (1965), *Factor Analysis of Data Matrices*, Holt, Rinehart and Winston, New York.
K. D. Ikramov and V. N. Chugunov (2000), 'Inverse matrix eigenvalue problems', *J. Math. Sci.* (New York) **98**, 51–136.
A. Iserles, H. Munthe-Kaas, S. P. Nørsett and A. Zanna (2000), Lie group methods, in *Acta Numerica*, Vol. 9, Cambridge University Press, pp. 1–151.
C. R. Johnson, T. J. Laffey and R. Loewy (1996), 'The real and the symmetric nonnegative inverse eigenvalue problems are different', *Proc. Amer. Math. Soc.* **124**, 3647–3651.

K. T. Joseph (1992), 'Inverse eigenvalue problem in structural design', *AIAA J.* **30**, 2890–2896.
N. K. Karmarkar and Y. N. Lakshman (1998), 'On approximate GCDs of univariate polynomials', *J. Symbolic Comput.* **26**, 653–666.
F. I. Karpelevič (1951), 'On the characteristic roots of matrices with nonnegative elements', *Izv. Akad. Nauk SSSR Ser. Mat.* **15**, 361–383. (In Russian.)
J. Kautsky and S. Elhay (1984), 'Gauss quadratures and Jacobi matrices for weight functions not of one sign', *Math. Comput.* **43**, 543–550.
J. Kautsky, N. K. Nichols and P. Van Dooren (1985), 'Robust pole assignment in linear state feedback', *Internat. J. Control* **41**, 1129–1155.
R. Knobel and J. R. McLaughlin (1994), 'A reconstruction method for a two-dimensional inverse eigenvalue problem', *Z. Angew. Math. Phys.* **45**, 794–826.
M. G. Kreĭn (1933), 'On the spectrum of a Jacobian matrix, in connection with the torsional oscillations of shafts', *Mat. Sbornik* **40**, 455–478.
J. D. Lambert (1991), *Numerical Methods for Ordinary Differential Systems: The Initial Value Problem*, Wiley, Chichester.
H. J. Landau (1994), 'The inverse eigenvalue problem for real symmetric Toeplitz matrices', *J. Amer. Math. Soc.* **7**, 749–767.
D. P. Laurie (1988), 'A numerical approach to the inverse Toeplitz eigenproblem', *SIAM J. Sci. Statist. Comput.* **8**, 410–405.
D. P. Laurie (1991), 'Solving the inverse eigenvalue problem via the eigenvector matrix', *J. Comput. Appl. Math.* **35**, 277–289.
D. P. Laurie (1997), 'Calculation of Gauss–Kronrod quadrature rules', *Math. Comput.* **66**, 1133–1145.
B. M. Levitan (1987), *Inverse Sturm–Liouville Problems*, VSP, Zeist. Translated from the Russian by O. Efimov.
C. K. Li and R. Mathias (2001), 'Construction of matrices with prescribed singular values and eigenvalues', *BIT* **41**, 115–126.
H. B. Li, P. Stoica and J. Li (1999), 'Computationally efficient maximum likelihood estimation of structured covariance matrices', *IEEE Trans. Signal Processing* **47**, 1314–1323.
N. Li (1997), 'A matrix inverse eigenvalue problem and its application', *Lin. Alg. Appl.* **266**, 143–152.
R. Loewy and D. London (1978), 'A note on an inverse problems for nonnegative matrices', *Lin. Multilin. Alg.* **6**, 83–90.
D. London and H. Minc (1972), 'Eigenvalues of matrices with prescribed entries', *Proc. Amer. Math. Soc.* **34**, 8–14.
J. R. McLaughlin and O. H. Hald (1995), 'A formula for finding a potential from nodal lines', *Bull. Amer. Math. Soc., New Series* **32**, 241–247.
J. R. McLaughlin, P. L. Polyakov and P. E. Sacks (1994), 'Reconstruction of a spherically symmetric speed of sound', *SIAM J. Appl. Math.* **54**, 1203–1223.
H. Minc (1988), *Nonnegative Matrices*, Wiley, New York.
L. Mirsky (1958), 'Matrices with prescribed characteristic roots and diagonal elements', *J. London Math. Soc.* **33**, 14–21.
J. E. Mottershead and M. I. Friswell, eds (2001), *Inverse Problem on Structural Dynamics* (Liverpool, 1999), Academic Press, Berlin. Special Issue of *Mechanical Systems and Signal Processing*.

M. Müller (1992), 'An inverse eigenvalue problem: Computing B-stable Runge–Kutta methods having real poles', *BIT* **32**, 676–688.

Y. Nievergelt (1997), 'Schmidt–Mirsky matrix approximation with linearly constrained singular values', *Lin. Alg. Appl.* **261**, 207–219.

P. Nylen (1999), 'Inverse eigenvalue problem: existence of special mass-damper-spring systems', *Lin. Alg. Appl.* **297**, 107–132.

P. Nylen and F. Uhlig (1997*a*), 'Inverse eigenvalue problem: existence of special spring-mass systems', *Inverse Problems* **13**, 1071–1081.

P. Nylen and F. Uhlig (1997*b*), Inverse eigenvalue problems associated with spring-mass systems, in *Proc. Fifth Conference of the International Linear Algebra Society* (Atlanta, GA, 1995), *Lin. Alg. Appl.* **254**, 409–425.

G. N. de Oliveira (1970), 'Note on an inverse characteristic value problem', *Numer. Math.* **15**, 345–347.

G. N. de Oliveira (1973*a*), 'Matrices with prescribed entries and eigenvalues, I', *Proc. Amer. Math. Soc.* **37**, 380–386.

G. N. de Oliveira (1973*b*), 'Matrices with prescribed entries and eigenvalues, II', *SIAM J. Appl. Math.* **24**, 414–417.

G. N. de Oliveira (1975), 'Matrices with prescribed entries and eigenvalues, III', *Arch. Math.* (Basel) **26**, 57–59.

G. N. de Oliveira (1983), 'Nonnegative matrices with prescribed spectrum', *Lin. Alg. Appl.* **54**, 117–121.

M. R. Osborne (1971), On the inverse eigenvalue problem for matrices and related problems for difference and differential equations, in *Conference on Applications of Numerical Analysis* (Dundee, 1971), Vol. 228 of *Lecture Notes in Mathematics*, Springer, Berlin, pp. 155–168.

J. Paine (1984), 'A numerical method for the inverse Sturm–Liouville problem', *SIAM J. Sci. Statist. Comput.* **5**, 149–156.

H. Park, L. Zhang and J. B. Rosen (1999), 'Low rank approximation of a Hankel matrix by structured total least norm', *BIT* **39**, 757–779.

B. N. Parlett (1998), *The Symmetric Eigenvalue Problem*, SIAM, Philadelphia, PA. Corrected reprint of the 1980 original.

H. Perfect (1953), 'Methods of constructing certain stochastic matrices', *Duke Math. J.* **20**, 395–404.

H. Perfect (1955), 'Methods of constructing certain stochastic matrices, II', *Duke Math. J.* **22**, 305–311.

J. D. Pryce (1993), *Numerical Solution of Sturm–Liouville Problems*, Oxford University Press, New York.

Y. Ram and G. M. L. Gladwell (1994), 'Constructing a finite element model of a vibratory rod from eigendata', *J. Sound Vib.* **169**, 229–237.

Y. M. Ram (1995), Pole-zero assignment of vibratory system by state feedback control, Technical report, University of Adelaide.

Y. M. Ram and J. Caldwell (1992), 'Physical parameters reconstruction of a free-free mass-spring system from its spectra', *SIAM J. Appl. Math.* **52**, 140–152.

Y. M. Ram and S. Elhay (1996), 'An inverse eigenvalue problem for the symmetric tridiagonal quadratic pencil with application to damped oscillatory systems', *SIAM J. Appl. Math.* **56**, 232–244.

M. S. Ravi, J. Rosenthal and X. A. Wang (1995), 'On decentralized dynamic pole placement and feedback stabilization', *IEEE Trans. Automat. Control* **40**, 1603–1614.

R. Reams (1996), 'An inequality for nonnegative matrices and the inverse eigenvalue problem', *Lin. Multilin. Alg.* **41**, 367–375.

A. Shapiro (1983), 'On the unsolvability of inverse eigenvalues problems almost everywhere', *Lin. Alg. Appl.* **49**, 27–31.

F. C. Silva (1987*a*), 'Matrices with prescribed characteristic polynomial and submatrices', *Portugal. Math.* **44**, 261–264.

F. C. Silva (1987*b*), 'Matrices with prescribed eigenvalues and principal submatrices', *Lin. Alg. Appl.* **92**, 241–250.

F. Y. Sing (1976), 'Some results on matrices with prescribed diagonal elements and singular values', *Canad. Math. Bull.* **19**, 89–92.

G. W. Soules (1983), 'Constructing symmetric nonnegative matrices', *Lin. Multilin. Alg.* **13**, 241–251.

L. Starek and D. J. Inman (1997), 'A symmetric inverse vibration problem for nonproportional underdamped systems', *Trans. ASME J. Appl. Mech.* **64**, 601–605.

L. Starek and D. J. Inman (2001), 'Symmetric inverse eigenvalue vibration problem and its applications', *Mechanical Systems and Signal Processing* **15**, 11–29.

L. Starek, D. J. Inman and A. Kress (1992), 'A symmetric inverse vibration problem', *Trans. ASME J. Vibration and Acoustics* **114**, 564–568.

T. J. Suffridge and T. L. Hayden (1993), 'Approximation by a Hermitian positive semidefinite Toeplitz matrix', *SIAM J. Matrix Anal. Appl.* **14**, 721–734.

H. R. Suleĭmanova (1949), 'Stochastic matrices with real characteristic numbers', *Doklady Akad. Nauk SSSR* (NS) **66**, 343–345.

J. G. Sun and Y. Qiang (1986), 'The unsolvability of inverse algebraic eigenvalue problems almost everywhere', *J. Comput. Math.* **4**, 212–226.

R. C. Thompson (1977), 'Singular values, diagonal elements, and convexity', *SIAM J. Appl. Math.* **32**, 39–63.

R. C. Thompson and P. McEnteggert (1968), 'Principal submatrices, II: The upper and lower quadratic inequalities.', *Lin. Alg. Appl.* **1**, 211–243.

F. Tisseur and K. Meerbergen (2001), 'The quadratic eigenvalue problem', *SIAM Review*. See also `http://www.ma.man.ac.uk/~ftisseur`.

W. F. Trench (1997), 'Numerical solution of the inverse eigenvalue problem for real symmetric Toeplitz matrices', *SIAM J. Sci. Comput.* **18**, 1722–1736.

D. W. Tufts and A. A. Shah (1993), 'Estimation of a signal wave-form from noisy data using low rank approximation to a data matrix', *IEEE Trans. Signal Processing* **41**, 1716–1721.

H. Weyl (1949), 'Inequalities between the two kinds of eigenvalues of a linear transformation', *Proc. Nat. Acad. Sci. USA* **35**, 408–411.

D. E. Williams and D. H. Johnson (1993), 'Robust estimation on structured covariance matrices', *IEEE Trans. Signal Processing* **41**, 2891–2906.

K. Wright (1992), 'Differential equations for the analytic singular value decomposition of a matrix', *Numer. Math.* **3**, 283–295.

Q. Wu (1990), Determination of the size of an object and its location in a cavity by eigenfrequency shifts, PhD thesis, University of Sydney.

S. F. Xu (1993), 'A stability analysis of the Jacobi matrix inverse eigenvalue problem', *BIT* **33**, 695–702.

S. F. Xu (1998), *An Introduction to Inverse Algebraic Eigenvalue Problems*, Peking University Press, and Friedr. Vieweg & Sohn, Braunschweig/Wiesbaden, Beijing.

E. M. E. Zayed (1993), 'An inverse eigenvalue problem for an arbitrary multiply connected bounded region: an extension to higher dimensions', *Internat. J. Math. Math. Sci.* **16**, 485–492.

H. Zha and Z. Zhang (1995), 'A note on constructing a symmetric matrix with specified diagonal entries and eigenvalues', *BIT* **35**, 448–452.

H. Zha and Z. Zhang (1999), 'Matrices with low-rank-plus-shift structure: partial SVD and latent semantic indexing', *SIAM J. Matrix Anal. Appl.* **21**, 522–536 (electronic).

L. A. Zhornitskaya and V. S. Serov (1994), 'Inverse eigenvalue problems for a singular Sturm–Liouville operator on $[0, 1]$', *Inverse Problems* **10**, 975–987.

S. Zhou and H. Dai (1991), *Daishu Tezhengzhi Fanwenti* (*The Algebraic Inverse Eigenvalue Problem*), Henan Science and Technology Press, Zhengzhou, China. (In Chinese.)